THE *HIGH SCHOOL* TRIGONOMETRY TUTOR®

SECOND EDITION

The Complete Study and Answer Guide to Any Textbook

- Includes every type of problem that may be assigned by your teacher or given on a test
- Makes Trigonometry **EASY** to learn, with fully explained answers to problems

REA *Research & Education Association*

THE HIGH SCHOOL TRIGONOMETRY TUTOR®

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WHAT THIS BOOK IS FOR

For as long as trigonometry and 11th year math has been taught in high schools, students have found this subject difficult to understand and learn. Despite the publication of hundreds of textbooks in this field, each one intended to provide an improvement over previous textbooks, students continue to remain perplexed, and the subject is often taken in class only to meet school/departmental requirements for a selected course of study.

In a study of the problem, REA found the following basic reasons underlying students' difficulties with trigonometry and 11th year math taught in schools:

(a) No systematic rules of analysis have been developed which students may follow in a step-by-step manner to solve the usual problems encountered. This results from the fact that the numerous different conditions and principles which may be involved in a problem, lead to many possible different methods of solution. To prescribe a set of rules to be followed for each of the possible variations, would involve an enormous number of rules and steps to be searched through by students, and this task would perhaps be more burdensome than solving the problem directly with some accompanying trial and error to find the correct solution route.

(b) Textbooks currently available will usually explain a given principle in a few pages written by a professional who has an insight in the subject matter that is not shared by students. The explanations are often written in an abstract manner which leaves the students confused as to the application of the principle. The explanations given are not sufficiently detailed and extensive to make the student aware of the wide range of applications and different aspects of the principle being studied. The numerous possible variations of principles and their applications are usually not discussed, and it is left for the students to discover these for themselves while doing the exercises. Accordingly, the average student is expected to

rediscover that which has been long known and practiced, but not published or explained extensively.

(c) The examples usually following the explanation of a topic are too few in number and too simple to enable the student to obtain a thorough grasp of the principles involved. The explanations do not provide sufficient basis to enable a student to solve problems that may be subsequently assigned for homework or given on examinations.

The examples are presented in abbreviated form which leaves out much material between steps, and requires that students derive the ommited material themselves. As a result, students find the examples difficult to understand--contrary to the purpose of the examples.

Examples are, furthermore, often worded in a confusing manner. They do not state the problem and then present the solution. Instead, they pass through a general discussion, never revealing what is to be solved for.

Examples, also, do not always include diagrams/graphs, wherever appropriate, and students do not obtain the training to draw diagrams or graphs to simplify and organize their thinking.

(d) Students can learn the subject only by doing the exercises themselves and reviewing them in class, to obtain experience in applying the principles with their different ramifications.

In doing the exercises by themselves, students find that they are required to devote considerbly more time to trigonometry and 11th year math than to other subjects of comparable credits, because they are uncertain with regard to the selection and application of the theorems and principles involved. It is also often necessary for students to discover those "tricks" not revealed in their texts (or review books), that make it possible to solve problems easily. Students must usually resort to methods of trial-and-error to discover these "tricks", and as a result they find that they may sometimes spend several hours in solving a single problem.

(e) When reviewing the exercises in classrooms, instructors

usually request students to take turns in writing solutions on the boards and explaining them to the class. Students often find it difficult to explain in a manner that holds the interest of the class, and enables the class to follow the material written on the boards. The remaining students seated in the class are, furthermore, too occupied with copying the material from the boards, to listen to the oral explanations and concentrate on the methods of solution.

This book is intended to aid students in trigonometry and 11th year math in overcoming the difficulties described, by supplying detailed illustations of the solution methods which are usually not apparent to students. The solution methods are illustrated by problems selected from those that are most often assigned for class work and given on examinations. The problems are arranged in order of complexity to enable students to learn and understand a particular topic by reviewing the problems in sequence. The problems are illustrated with detailed step-by-step explanations, to save the students the large amount of time that is often needed to fill in the gaps that are usually found between steps of illustrations in textbooks or review/outline books.

The staff of REA considers trigonometry and 11th year math a subject that is best learned by allowing students to view the methods of analysis and solution techniques themselves. This approach to learning the subject matter is similar to that practiced in various scientific laboratories, particularly in the medical fields.

In using this book, students may review and study the illustrated problems at their own pace; they are not limited to the time allowed for explaining problems on the board in class.

When students want to look up a particular type of problem and solution, they can readily locate it in the book by referring to the index which has been extensively prepared. It is also possible to locate a particular type of problem by glancing at just the material within the boxed portions. To facilitate rapid scanning of the problems, each problem has a heavy border

around it. Furthermore, each problem is identified with a number immediately above the problem at the right-hand margin.

To obtain maximum benefit from the book, students should familiarize themselves with the section, "How To Use This Book," located in the front pages.

To meet the objectives of this book, staff members of REA have selected problems usually encountered in assignments and examinations, and have solved each problem meticulously to illustrate the steps which are usually difficult for students to comprehend. Special gratitude is expressed to them for their efforts in this area, as well as to the numerous contributors who devoted brief periods of time to this work.

Gratitude is also expressed to the many persons involved in the difficult task of typing the manuscript with its endless changes, and to the REA art staff who prepared the numerous detailed illustrations together with the layout and physical features of the book.

The difficult task of coordinating the efforts of all persons was carried out by Carl Fuchs. His conscientious work deserves much appreciation. He also trained and supervised art and production personnel in the preparation of the book for printing.

Finally, special thanks are due to Helen Kaufmann for her unique talents to render those difficult border-line decisions and constructive suggestions related to the design and organization of the book.

Max Fogiel, PH. D.
Program Director

HOW TO USE THIS BOOK

This book can be an invaluable aid to students in trigonometry and 11th year math as a supplement to their textbooks. The book is subdivided into 12 chapters, each dealing with a separate topic. The subject matter is developed beginning with basic trigonometric relations and functions and extending through linear equations, quadratic equations, logarithms, exponentials and applied problems.

TO LEARN AND UNDERSTAND A TOPIC THOROUGHLY

1. Refer to your class text and read there the section pertaining to the topic. You should become acquainted with the principles discussed there. These principles, however, may not be clear to you at that time.

2. Then locate the topic you are looking for by referring to the "Table of Contents" in front of this book.

3. Turn to the page where the topic begins and review the problems under each topic, in the order given. For each topic, the problems are arranged in order of complexity, from the simplest to the more difficult. Some problems may appear similar to others, but each problem has been selected to illustrate a different point or solution method.

To learn and understand a topic thoroughly and retain its contents, it will generally be necessary for students to review the problems several times. Repeated review is essential in order to gain experience in recognizing the principles that should be applied, and to select the best solution technique.

TO FIND A PARTICULAR PROBLEM

To locate one or more problems related to a particular subject matter, refer to the index. In using the index, be certain to note that the numbers given there refer to problem

numbers, not to page numbers. This arrangement of the index is intended to facilitate finding a problem more rapidly, since two or more problems may appear on a page.

If a particular type of problem cannot be found readily, it is recommended that the student refer to the "Table of Contents" in the front pages, and then turn to the chapter which is applicable to the problem being sought. By scanning or glancing at the material that is boxed, it will generally be possible to find problems related to the one being sought, without consuming considerable time. After the problems have been located, the solutions can be reviewed and studied in detail. For this purpose of locating problems rapidly, students should acquaint themselves with the organization of the book as found in the "Table of Contents".

In preparing for an exam, locate the topics to be covered on the exam in the "Table of Contents," and then review the problems under those topics several times. This should equip the student with what might be needed for the exam.

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CHAPTER 1

BASIC MATH

FUNDAMENTAL ALGEBRAIC LAWS AND OPERATIONS

• PROBLEM 1-1

Show that $(-2) + (-3) = -5$.

Solution: This small problem illustrates some of the basic ideas involved in mathematical proof. We know that $(-2) + (-3)$ is an integer because the integers are closed under addition. To show that this integer is -5, we ask ourselves what property is characteristic of -5.

$$5 + (-5) = 0,$$

by the additive inverse property, $a + (-a) = 0$. Moreover, -5 is the only number which when added to 5 gives 0; for if $(5 + b) = 0$, by the additive identity, $a = a + 0$, $-5 = -5 + 0 = -5 + (5 + b)$, by our hypothesis, $5 + b = 0$

$= (-5 + 5) + b$ by associative law of addition $a + (b+c) = (a+b)+c$.

$= 0 + b$ by additive inverse property, $a + (-a) = 0$

$= b + 0$ by commutative law of addition, $a + b = b + a$

$= b$ by additive identity, $a + 0 = a$

Thus, $-5 = b$, proving that -5 is the only number which when added to 5 gives 0.

We therefore see that $(-2) + (-3) = -5$ if and only

if $5 + [(-2) + (-3)] = 0$. We show below that this sum is zero.

$$5 + [(-2) + (-3)] = [3 + 2] + [(-2) + (-3)]$$
$$= 3 + \{2 + [(-2) + (-3)]\},$$

by associative law of addition, $(a + b) + c = a + (b + c)$;
$$= 3 + [0 + (-3)],$$

by additive inverse property, $a + (-a) = 0$; $= 3 + [(-3) + 0]$, by commutative law of addition, $a + b = b + a$;

$$= 3 + (-3),$$

by additive identity property, $a + 0 = a$; $= 0$, by additive inverse property, $a + (-a) = 0$.

Thus we have shown (a) $5 + (-5) = 0$

(b) (-5) is the only number which when added to 5 equals 0.

(c) $5 + (-5) = 0 = 5 + [(-2) + (-3)]$ and therefore

$$(-5) = (-2) + (-3),$$

completing our proof.

• PROBLEM 1-2

Evaluate $2 - \{5 + (2 - 3) + [2 - (3 - 4)]\}$

Solution: When working with a group of nested parentheses, we evaluate the innermost parenthesis first.

Thus, $2 - \{5 + (2 - 3) + [2 - (3 - 4)]\}$
$$= 2 - \{5 + (2 - 3) + [2 - (-1)]\}$$
$$= 2 - \{5 + (-1) + [2 + 1]\}$$
$$= 2 - \{5 + (-1) + 3\}$$
$$= 2 - \{4 + 3\}$$
$$= 2 - 7$$
$$= -5.$$

• PROBLEM 1-3

Simplify $4[-2(3 + 9) \div 3] + 5$.

Solution: To simplify means to find the simplest expression. We perform the operations within the innermost grouping symbols first. That is $3 + 9 = 12$.

Thus, $4[-2(3 + 9) \div 3] + 5 = 4[-2(12) \div 3] + 5$

Next we simplify within the brackets:

$$= 4[-24 \div 3] + 5$$
$$= 4 \cdot (-8) + 5$$

We now perform the multiplication, since multiplication is done before addition:

$$= -32 + 5$$
$$= -27$$

Hence, $4[-2(3 + 9) \div 3] + 5 = -27$.

• PROBLEM 1-4

Is the set of all natural numbers from 1 to 10 a closed system under addition?

Solution: For $\{1,2,3,\ldots 10\}$ to be closed with respect to addition, the sum of any two numbers in this set must also be a member of this set. The set of all natural numbers from 1 to 10, inclusive, is therefore not a closed system under addition for it would not be correct to say that given any two numbers in the set there is a number in the set called their sum. For instance, 4 and 7 are in the set but their sum, 11, is not.

• PROBLEM 1-5

Simplify
$$8x^2 - \left[7x - \left(x^2 - x + 5y\right)\right] + \left(2x - 3y\right).$$

Solution: Where a succession of arithmetic operations is involved, appropriate grouping symbols indicate clearly how these algebraic operations are to be performed; that is, we perform them before other operations. In this problem we also have grouping symbols within grouping symbols. Therefore, we perform the operation in the innermost parentheses first. Hence, multiply the terms in the parentheses by minus one in order to remove the parentheses. Furthermore, they can be removed from the last two terms.

$$8x^2 - \left[7x - (x^2 - x + 5y)\right] + (2x - 3y) = 8x^2 - \left[7x - x^2 + x - 5y\right] + 2x - 3y$$

Remove the brackets by multiplying the terms inside by minus one. Then,

$$8x^2 - \left[7x - (x^2 - x + 5y)\right] + (2x - 3y) = 8x^2 - 7x + x^2 - x + 5y + 2x - 3y$$

Now group like terms. Then perform the indicated operations from left to right. Thus, we obtain:

$$8x^2 - \left[7x - (x^2 - x + 5y)\right] + (2x - 3y) = 8x^2 + x^2 - 7x - x + 2x + 5y - 3y$$

$$= 9x^2 - 6x + 2y.$$

In this example, we have found the algebraic sum of these three quantities: $8x^2$ and $-\left[7x - (x^2 - x + 5y)\right]$ and $(2x - 3y)$.

• PROBLEM 1-6

> Use the field properties to derive the equation $x = 5$ from the equation $5x - 3 = 2(x + 6)$.

Solution:

$5x - 3 = 2(x + 6)$	Given
$5x - 3 = 2x + 12$	distributive property of multiplication over addition
$(5x - 3) + (-2x) = 2x + 12 + (-2x)$	Additive Property $(-2x)$
$3x - 3 = 12$	Simplifying
$(3x - 3) + 3 = 12 + 3$	additive property $(+3)$
$3x = 15$	Simplifying
$\frac{1}{3}\cdot(3x) = \frac{1}{3}\cdot 15$	Multiplicative Property$(\frac{1}{3})$
$x = 5$	Simplifying

We could also derive $5x - 3 = 2(x + 6)$ from $x = 5$ by reversing the steps in the solution. Let us see if 5 will make the equation $5x - 3 = 2(x + 6)$ true.

$$5(5) - 3 \stackrel{?}{=} 2(5 + 6)$$

$$22 = 22$$

Two equations are equivalent if and only if they have the same solution set. Since $5x - 3 = 2(x + 6)$ and $x = 5$ have the same solution set, $\{5\}$, the two equations are equivalent.

• PROBLEM 1-7

> Evaluate $p = \dfrac{(a - b)(ab + c)}{(cb - 2a)}$
>
> when $a = +2$, $b = -\frac{1}{2}$, and $c = -3$.

Solution: Inserting the given values of a, b, and c

$$p = \frac{[+2 - (-\frac{1}{2})][(+2)(-\frac{1}{2}) + (-3)]}{[(-3)(-\frac{1}{2}) - 2(+2)]}$$

$$= \frac{[+2 + \frac{1}{2}][-1 - 3]}{[+1\frac{1}{2} - 4]}$$

$$= \frac{(2\frac{1}{2})(-4)}{-(2\frac{1}{2})}$$

The $2\frac{1}{2}$ in the numerator cancels the $2\frac{1}{2}$ in the denominator.

$$p = \frac{-4}{-1}$$

Multiplying numerator and denominator by -1

$$p = \frac{+4}{+1}$$

$$p = +4.$$

LEAST COMMON MULTIPLE; GREATEST COMMON DIVISOR

• PROBLEM 1-8

> Find the least common multiple (lcm) of 15 and 18.

Solution: Some of the integers divisible by 15 are

$$15, 30, 45, 60, 75, 90, 105, \ldots$$

Some of the integers divisible by 18 are

$$18, 36, 54, 72, 90, 108, \ldots$$

The smallest positive integer divisible by both 15 and 18 is 90. Thus,

$$\text{lcm}\{15, 18\} = 90$$

Another method for finding $\text{lcm}\{15,18\}$ is the following:

Factor 15 and 18 into their prime factors.

$$15 = 3\cdot 5$$

$$18 = 2\cdot 3\cdot 3$$

Now, take the different factors of the two numbers and multiply them together. The exponent to be used for each factor is the highest number of times that the factor appears in either number (15 or 18). The product obtained will be the $\text{lcm}\{15,18\}$. Hence:

$$\text{lcm}\{15,18\} = 2^1 \cdot 3^2 \cdot 5^1 = 2(9)(5) = 90.$$

• **PROBLEM 1-9**

Find the greatest common divisor $\{15,28\}$.

Solution: If 15 and 28 are factored completely into their respective prime factors, $15 = 3 \cdot 5$ and $28 = 2 \cdot 2 \cdot 7$

Since 1 divides every integer, and since 15 and 28 possess no common prime factors, it follows that

$$\gcd\{15,28\} = 1.$$

If the gcd of two integers is 1, then the two integers are said to be relatively prime. Since $\gcd\{15,28\} = 1$, the integers 15 and 28 are relatively prime.

• **PROBLEM 1-10**

If $a = \{1, 2, 3, 4, 5)$ and $b = \{2, 3, 4, 5, 6\}$, find $a \cap b$.

Solution: The intersection of two sets a and b is the set of all elements that belong to both a and b; that is, all elements common to a and b. In this problem, if

$$a = \{1, 2, 3, 4, 5\} \text{ and } b = \{2, 3, 4, 5, 6\},$$

then

$$a \cap b = \{2, 3, 4, 5\}.$$

ABSOLUTE VALUES

• **PROBLEM 1-11**

Solve for x when $|x - 7| = 3$.

Solution: This equation, according to the definition of absolute value, expresses the conditions that $x - 7$ must be 3 or -3, since in either case the absolute value is 3. If $x - 7 = 3$, we have $x = 10$; and if $x - 7 = -3$, we have $x = 4$. We see that there are two values of x which solve the equation.

• **PROBLEM 1-12**

Solve for x when $|3x + 2| = 5$.

Solution: First we write expressions which replace the absolute symbols in forms of equations that can be manipulated algebraically. Thus this equation will be satisfied if either

$$3x + 2 = 5 \quad \text{or} \quad 3x + 2 = -5.$$

Considering each equation separately, we find

$$x = 1 \quad \text{and} \quad x = -\frac{7}{3}.$$

Accordingly, the given equation has two solutions.

• **PROBLEM 1-13**

Solve for x in $|2x - 6| = |4 - 5x|$.

Solution: There are four possibilities here. $2x - 6$ and $4 - 5x$ can be either positive or negative. Therefore,

$$2x - 6 = 4 - 5x \qquad (1)$$

$$-(2x - 6) = 4 - 5x \qquad (2)$$

and

$$2x - 6 = -(4 - 5x) \qquad (3)$$

$$-(2x - 6) = -(4 - 5x) \qquad (4)$$

Equations (2) and (3) result in the same solution, as do equations (1) and (4). Therefore it is necessary to solve only for equations (1) and (2). This gives:

$$x = \frac{10}{7}, -\frac{2}{3}.$$

OPERATIONS WITH FRACTIONS

• **PROBLEM 1-14**

Simplify $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{6}}$.

Solution: $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{6}}$ means $\left(\frac{1}{2} + \frac{1}{3}\right) \div \frac{1}{6}$

Since division by a fraction is equivalent to multiplication by its reciprocal:

$$\left(\frac{1}{2} + \frac{1}{3}\right) \div \frac{1}{6} = \left(\frac{1}{2} + \frac{1}{3}\right) \times 6$$

By the distributive law:

$$\left(\frac{1}{2}+\frac{1}{3}\right) \times 6 = \left(\frac{1}{2}\right)(6) + \left(\frac{1}{3}\right)(6)$$
$$= \frac{6}{2} + \frac{6}{3}$$
$$= 3 + 2$$
$$= 5$$

• PROBLEM 1-15

Simplify the following expression: $1 - \dfrac{1}{2 - \frac{1}{3}}$.

Solution: In order to combine the denominator, $2 - \frac{1}{3}$, we must convert 2 into thirds. $2 = 2 \cdot 1 = 2 \cdot \frac{3}{3} = \frac{6}{3}$. Thus

$$1 - \frac{1}{2 - \frac{1}{3}} = 1 - \frac{1}{\frac{6}{3} - \frac{1}{3}} = 1 - \frac{1}{\frac{5}{3}}$$

Since division by a fraction is equivalent to multiplication by that fraction's reciprocal

$$1 - \frac{1}{\frac{5}{3}} = 1 - (1)\left(\frac{3}{5}\right) = 1 - \frac{3}{5} = \frac{5}{5} - \frac{3}{5} = \frac{2}{5}$$

Therefore, $$1 - \frac{1}{2 - \frac{1}{3}} = \frac{2}{5}.$$

• PROBLEM 1-16

Simplify $\dfrac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{4} - \frac{1}{3}}$.

Solution: A first method is to just add the terms in the numerator and denominator. Since 6 is the least common denominator of the numerator, $\left(\frac{2}{3} + \frac{1}{2}\right)$, we convert $\frac{2}{3}$ and $\frac{1}{2}$ into sixths:

$\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{2}{2} = \frac{4}{6}$ and $\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$

Therefore $\frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}$

Since 12 is the least common denominator of the denominator, $\left(\frac{3}{4} - \frac{1}{3}\right)$, we convert $\frac{3}{4}$ and $\frac{1}{3}$ into twelfths:

$\frac{3}{4} = \frac{3}{4} \cdot 1 = \frac{3}{4} \cdot \frac{3}{3} = \frac{9}{12}$ and $\frac{1}{3} = \frac{1}{3} \cdot 1 = \frac{1}{3} \cdot \frac{4}{4} = \frac{4}{12}$

Therefore $\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12} = \frac{5}{12}$

Thus, $$\frac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{4} - \frac{1}{3}} = \frac{\frac{7}{6}}{\frac{5}{12}}$$

Division by a fraction is equivalent to multiplication by the reciprocal hence $\dfrac{\frac{7}{6}}{\frac{5}{12}} = \frac{7}{6} \cdot \frac{12}{5}$

Cancelling 6 from the numerator and denominator:

$$= \frac{7}{1} \cdot \frac{2}{5} = \frac{14}{5}$$

A second method is to multiply both numerator and denominator by the least common denominator of the entire fraction. Since we have already seen that L.C.D. of the numerator is 6 and the L.C.D. of the denominator is 12, and 12 is divisible by 6, we use 12 as the L.C.D. of the entire fraction. Thus

$$\frac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{4} - \frac{1}{3}} = \frac{12\left(\frac{2}{3} + \frac{1}{2}\right)}{12\left(\frac{3}{4} - \frac{1}{3}\right)}$$

Distribute:

$$= \frac{12\left(\frac{2}{3}\right) + 12\left(\frac{1}{2}\right)}{12\left(\frac{3}{4}\right) - 12\left(\frac{1}{3}\right)}$$

$$= \frac{4 \cdot 2 + 6}{3 \cdot 3 - 4} = \frac{8 + 6}{9 - 4} = \frac{14}{5}$$

• PROBLEM 1-17

If $a = 4$ and $b = 7$ find the value of $\dfrac{a + \frac{a}{b}}{a - \frac{a}{b}}$.

Solution: By substitution, $\dfrac{a + \frac{a}{b}}{a - \frac{a}{b}} = \dfrac{4 + \frac{4}{7}}{4 - \frac{4}{7}}$.

In order to combine the terms we convert 4 into sevenths:

$$4 = 4 \cdot 1 = 4 \cdot \frac{7}{7} = \frac{28}{7}.$$

Thus, we have:

$$\frac{\frac{28}{7}+\frac{4}{7}}{\frac{28}{7}-\frac{4}{7}} = \frac{\frac{32}{7}}{\frac{24}{7}} .$$

Dividing by $\frac{24}{7}$ is equivalent to multiplying by $\frac{7}{24}$. Therefore,

$$\frac{4+\frac{4}{7}}{4-\frac{4}{7}} = \frac{32}{7} \cdot \frac{7}{24}$$

Now, the 7 in the numerator cancels with the 7 in the denominator.

Thus, we obtain: $\frac{32}{24}$, and dividing numerator and denominator by 8,

we obtain: $\frac{4}{3}$.

Therefore $\dfrac{a+\frac{a}{b}}{a-\frac{a}{b}} = \frac{4}{3}$ when a = 4 and b = 7.

• PROBLEM 1-18

Combine and simplify $1 + \dfrac{1}{1+\frac{1}{1-x}}$.

Solution: First combine the terms in the denominator. Recall $1 = (1-x)/(1-x)$. Thus,

$$1 + \frac{1}{1+\frac{1}{1-x}} = 1 + \frac{1}{\frac{1-x}{1-x}+\frac{1}{1-x}}$$

$$= 1 + \frac{1}{\frac{1-x+1}{1-x}}$$

$$= 1 + \frac{1}{\frac{2-x}{1-x}}$$

Division by a fraction is equivalent to multiplication by its reciprocal, thus

$$= 1 + 1 \cdot \frac{(1-x)}{(2-x)}$$

$$= 1 + \frac{1-x}{2-x}$$

Recall $1 = \frac{2-x}{2-x}$, therefore,

$$1 + \frac{1}{1+\frac{1}{1-x}} = \frac{2-x}{2-x} + \frac{1-x}{2-x}$$

$$= \frac{2-x+1-x}{2-x}$$

$$= \frac{3-2x}{2-x} .$$

• PROBLEM 1-19

Simplify $\dfrac{x+\frac{1}{y}}{x-\frac{1}{y}}$.

Solution: Obtain the least common denominator, ℓ.c.d., of the two terms in the numerator and of those that appear in the denominator. This is done by writing down the different factors that appear in the denominators of the terms. The exponent to be used for each factor is the smallest number of times that the factor appears in either of the denominators of the terms. Hence, the ℓ.c.d. of the two terms in the denominator $= (1)^1(y)^1 = 1y = y$. Also, the ℓ.c.d. of the two terms in the numerator is obtained in the same way. Therefore, the ℓ.c.d. of the two terms in the numerator $= (1)^1(y)^1 = 1y = y$. Therefore:

$$\frac{x+\frac{1}{y}}{x-\frac{1}{y}} = \frac{\frac{y(x)}{y}+\frac{1}{y}}{\frac{y(x)}{y}-\frac{1}{y}} = \frac{\frac{yx+1}{y}}{\frac{yx-1}{y}}$$

Division is the same as multiplying the numerator by the multiplicative inverse of the denominator. (The multiplicative inverse of a number a is the number n, such that $a \cdot n = 1$. This number n is $1/a$. Hence, $a \cdot 1/a = 1$.) Therefore, the multiplicative inverse of

$$\frac{yx-1}{y} \text{ is } \frac{y}{yx-1}.$$

Hence,

$$\frac{x+\frac{1}{y}}{x-\frac{1}{y}} = \frac{\frac{yx+1}{y}}{\frac{yx-1}{y}} = \left(\frac{yx+1}{\cancel{y}}\right)\left(\frac{\cancel{y}}{yx-1}\right) = \frac{yx+1}{yx-1} .$$

• PROBLEM 1-20

Simplify this expression:

$$\frac{\frac{3}{x}-\frac{2}{y}}{\frac{5}{x}+\frac{6}{y}}$$

Solution: There are two ways to approach this problem. One is to consider it as a division problem:

$$\left(\frac{3}{x} - \frac{2}{y}\right) \div \left(\frac{5}{x} + \frac{6}{y}\right) .$$

Use the least common denominator xy :

$$= \left(\frac{3y}{xy} - \frac{2x}{xy}\right) \div \left(\frac{5y}{xy} + \frac{6x}{xy}\right) .$$

Combine fractions: $$= \frac{3y-2x}{xy} \div \frac{5y+6x}{xy}.$$

Dividing by a fraction is equivalent to multiplying by its reciprocal:

$$= \left(\frac{3y-2x}{xy}\right) \cdot \left(\frac{xy}{5y+6x}\right) .$$

Cancelling out xy : $$= \frac{3y-2x}{5y+6x} .$$

The second approach is to multiply both numerator and denominator by xy; this is equivalent to multiplying the fraction by 1:

$$\frac{\frac{3}{x} - \frac{2}{y}}{\frac{5}{x} + \frac{6}{y}} \cdot \frac{xy}{xy} = \frac{3y - 2x}{5y + 6x} .$$

• PROBLEM 1-21

Combine $\frac{1}{6x} + \frac{1}{3y} - \frac{3x + 2y}{12xy}$ into a single fraction.

Solution: Since both $6x$ and $3y$ are factors of $12xy$, the least common denominator (the LCD) of the given fractions is $12xy$. Thus, we wish to convert the given fractions to equal fractions having $12xy$ as a denominator. We can accomplish this by multiplying each member of the first fraction by $2y$ and each member of the second by $4x$. We thereby obtain

$$\frac{1}{6x} + \frac{1}{3y} - \frac{3x + 2y}{12xy} = \frac{2y \cdot 1}{2y(6x)} + \frac{4x \cdot 1}{4x(3y)} - \frac{3x + 2y}{12xy}$$

$$= \frac{2y}{12xy} + \frac{4x}{12xy} - \frac{3x + 2y}{12xy}$$

$$= \frac{2y + 4x - (3x + 2y)}{12xy}$$

$$= \frac{2y + 4x - 3x - 2y}{12xy}$$

$$= \frac{2y + x - 2y}{12xy}$$

$$= \frac{2y - 2y + x}{12xy}$$

$$= \frac{x}{12xy} .$$

Cancelling x from numerator and denominator,

$$= \frac{1}{12y}$$

Thus, $$\frac{1}{6x} + \frac{1}{3y} - \frac{3x + 2y}{12xy} = \frac{1}{12y} .$$

• PROBLEM 1-22

Simplify: $$\frac{\frac{1}{a} - \frac{1}{a - b}}{\frac{1}{a} + \frac{1}{a - b}}$$

Solution: The Lowest Common Multiple (L.C.M) of the denominators is $a(a - b)$. Since $\frac{a(a-b)}{a(a-b)} = 1$, multiply numerator and denominator by $a(a - b)$. Thus, we have:

$$\frac{a(a - b)\left(\frac{1}{a} - \frac{1}{a - b}\right)}{a(a - b)\left(\frac{1}{a} + \frac{1}{a - b}\right)} .$$

Distribute $a(a - b)$ in numerator and denominator:

$$\frac{a(a - b)\left(\frac{1}{a}\right) - a(a - b)\left(\frac{1}{a - b}\right)}{a(a - b)\left(\frac{1}{a}\right) + a(a - b)\left(\frac{1}{a - b}\right)} .$$

Perform the multiplication:

$$\frac{\frac{a}{a}(a - b) - a\left(\frac{a - b}{a - b}\right)}{\frac{a}{a}(a - b) + a\left(\frac{a - b}{a - b}\right)}$$

Since $\frac{a}{a} = 1$ and $\frac{a - b}{a - b} = 1$, we have: $\frac{(a - b) - a}{(a - b) + a}$.

Using Associative and Commutative laws of addition, we obtain:

$$\frac{\frac{1}{a} - \frac{1}{a - b}}{\frac{1}{a} + \frac{1}{a - b}} = \frac{(a - a)- b}{(a + a)- b} = \frac{-b}{2a - b} .$$

● **PROBLEM 1-23**

Simplify $\dfrac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{y}{x+y}-\dfrac{x}{x-y}}$.

Solution: To eliminate the fractions in this expression, we multiply numerator and denominator by the least common multiple (L.C.M.), the expression of lowest degree into which each of the original expressions can be divided without a remainder. The L.C.M. is the product obtained by taking each factor to the highest degree. In our case the L.C.M. is $(x + y)(x - y)$. Thus, multiplying numerator and denominator by $(x + y)(x - y)$,

$$\frac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{y}{x+y}-\dfrac{x}{x-y}} = \frac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{y}{x+y}-\dfrac{x}{x-y}} \cdot \frac{(x+y)(x-y)}{(x+y)(x-y)} .$$

Distributing, $$= \frac{\left(\dfrac{x}{x+y}\right)(x+y)(x-y)+\left(\dfrac{y}{x-y}\right)(x+y)(x-y)}{\left(\dfrac{y}{x+y}\right)(x+y)(x-y)-\left(\dfrac{x}{x-y}\right)(x+y)(x-y)}$$

Cancelling like terms, $$= \frac{x(x-y)+y(x+y)}{y(x-y)-x(x+y)}$$

Distributing, $$= \frac{x^2 - xy + yx + y^2}{yx - y^2 - x^2 - xy}$$

Using the commutative law, $$= \frac{x^2 - xy + xy + y^2}{xy - y^2 - x^2 - xy}$$

Combining terms, $$= \frac{x^2 + y^2}{-x^2 - y^2}$$

Factoring (-1) from the denominator $$= \frac{x^2 + y^2}{(-1)(x^2 + y^2)}$$

Cancelling $x^2 + y^2$, $$= \frac{1}{-1}$$

$$= -1.$$

Thus, $$\frac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{y}{x+y}-\dfrac{x}{x-y}} = -1.$$

● **PROBLEM 1-24**

A) If $x = \dfrac{c - ab}{a - b}$, find the value of the expression $a(x + b)$.

B) Also, if $x = \dfrac{c - ab}{a - b}$, find the value of the expression $bx + c$.

Solution: A) Substituting $x = \dfrac{c - ab}{a - b}$ for x in the expression $a(x + b)$,

$$a(x + b) = a\left(\frac{c - ab}{a - b} + b\right) \qquad (1)$$

Obtaining a common denominator of $a - b$ for the two terms in parenthesis; equation (1) becomes:

$$a(x + b) = a\left[\frac{c - ab}{a - b} + \frac{(a-b)b}{a - b}\right]$$

Distributing the numerator of the second term in brackets:

$$a(x + b) = a\left[\frac{c-ab}{a-b} + \frac{ab-b^2}{a-b}\right] = a\left[\frac{c-\cancel{ab}+\cancel{ab}-b^2}{a-b}\right]$$

$$= a\left[\frac{c-b^2}{a-b}\right]$$

$$a(x + b) = \frac{a(c-b^2)}{a-b}$$

B) Substituting $x = \dfrac{c-ab}{a-b}$ for x in the expression $bx + c$,

$$bx + c = b\left(\frac{c-ab}{a-b}\right) + c$$

$$= \frac{b(c-ab)}{a-b} + c \qquad (2)$$

Obtaining a common denominator of $a - b$ for the two terms on the right side of equation (2):

$$bx + c = \frac{b(c-ab)}{a-b} + \frac{(a-b)c}{a-b}$$

Distributing the numerator of each term on the right side:

$$bx + c = \frac{\cancel{bc}-ab^2}{a-b} + \frac{ac-\cancel{bc}}{a-b}$$

$$= \frac{\cancel{bc}-ab^2+ac-\cancel{bc}}{a-b} = \frac{-ab^2+ac}{a-b}$$

$$= \frac{ac-ab^2}{a-b}$$

Factoring out the common factor of a from the numerator of the right side:

$$bx + c = \frac{a(c-b^2)}{a-b}$$

• PROBLEM 1-25

Write $\frac{2}{7}$ as a repeating decimal.

Solution: To write a fraction as a repeating decimal divide the numerator by the denominator, until a pattern of repeated digits appears.

$$2 \div 7 = .285714285714...$$

Identify the entire portion of the decimal which is repeated. The repeating decimal can then be written in the shortened form:

$$\frac{2}{7} = .\overline{285714}$$

• PROBLEM 1-26

Find the common fraction form of the repeating decimal 0.4242....

Solution: Let x represent the repeating decimal.

$$x = 0.4242...$$

$$100x = 42.42... \quad \text{by multiplying by 100}$$

$$x = 0.42...$$

$$99x = 42 \qquad (1) \text{ by subtracting x from 100x}$$

Divide both sides of equation (1) by 99.

$$\frac{99x}{99} = \frac{42}{99}$$

$$x = \frac{42}{99} = \frac{14}{33}$$

The repeating decimal of this example had 2 digits that repeated. The first step in the solution was to multiply both sides of the original equation by the 2nd power of 10 or 10^2 or 100. If there were 3 digits that repeated, the first step in the solution would be to multiply both sides of the original equation by the 3rd power of 10 or 10^3 or 1000.

• PROBLEM 1-27

Find $0.25\overline{25}$ as a quotient of integers.

Solution: Let $x = 0.25\overline{25}$. (1) Multiply both sides of this equation by 100:

$$100x = 100(0.25\overline{25})$$

Multiplying by 100 is equivalent to moving the decimal two places to the right, and since the digits 25 are repeated we have:

$$100x = 25.25\overline{25} \qquad (2)$$

Now subtract equation (1) from equation (2):

$$100x = 25.25\overline{25}$$

$$-\ x = 0.25\overline{25}$$

$$99x = 25.0000$$

or $\quad 99x = 25 \qquad (3)$

(Note that this operation eliminates the decimal)
Dividing both sides of equation (3) by 99:

$$\frac{99x}{99} = \frac{25}{99}$$

$$x = \frac{25}{99}$$

Therefore,

$$0.25\overline{25} = x = \frac{25}{99}$$

Also, note that the given repeating decimal, $0.25\overline{25}$, was multiplied by 100 or 10^2, where the power of 10 (which is 2) is the same as the number of repeating digits (namely, 2) for this problem. In general, for problems of this type, if the repeating decimal has n repeating digits, then the repeating decimal should be multiplied by 10^n.

• PROBLEM 1-28

Write the repeating decimal $14.\overline{23}$ as a quotient of two integers, $\frac{p}{q}$.

Solution: Let $x = 14.\overline{23}$. (1) Multiply both sides of this equation by 100:

$$100x = 100(14.\overline{23})$$

$$100x = 1423.\overline{23} \qquad (2)$$

Subtract equation (1) from equation (2):

$$\begin{array}{rl} 100x = & 1423.\overline{23} \\ - \quad x = & 14.\overline{23} \\ \hline 99x = & 1409.00 \end{array}$$

or $$99x = 1409 \qquad (3)$$

(Note that this operation eliminates the decimal.) Dividing both sides of equation (3) by 99:

$$\frac{99x}{99} = \frac{1409}{99}$$

$$x = \frac{1409}{99}.$$

Therefore,

$$14.\overline{23} = x = \frac{1409}{99}.$$

Also, note that the given repeating decimal, $14.\overline{23}$, was multiplied by 100 or 10^2, where the power of 10 (which is 2) is the same as the number of repeating digits (namely, 2) for this problem. In general, for problems of this type, if the repeating decimal has n repeating digits, then the repeating decimal should be multiplied by 10^n.

ALGEBRAIC ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION

• PROBLEM 1-29

Evaluate the expression $3x^2y - 2xy^2z + xyz$ when $x = 2$, $y = -1$, and $z = 3$.

Solution: There are two ways to find the value.

(a) We can simplify $3x^2y - 2xy^2z + xyz$ and substitute. Simplifying can be done by applying the distributive property since each term of the expression has a common factor of xy. Thus, we have

$$(3x - 2yz + z)xy$$

We now substitute in the expression:

$$[3(2) - 2(-1)(3) + 3](2)(-1) = (6 + 6 + 3)(-2) = -30.$$

(b) We can substitute in the original expression:

$$3(2)^2(-1) - 2(2)(-1)^2(3) + (2)(-1)(3) = -12 - 12 - 6 = -30.$$

In the example above we found that both $(3x - 2yz + z)xy$ and $3x^2y - 2xy^2z + xyz$ have the same value when $x = 2$, $y = -1$, and $z = 3$. It is apparent that both expressions will have equal values for any set of replacements of the variable; they are called equivalent expressions.

• PROBLEM 1-30

Combine the expressions $2a - 5b - c$ and $8a + 4b - 3c$.

Solution: Whenever two expressions are combined, those two expressions should be added. Therefore, the two expressions $2a-5b-c$ and $8a+4b-3c$ to be combined will be added. $(2a-5b-c) + (8a+4b-3c)$ is the problem. Place all similiar terms together; that is, place the a terms together, the b terms together and the c terms together. Therefore:

$$\begin{aligned}(2a - 5b - c) + (8a + 4b - 3c) &= (2a + 8a) + (-5b + 4b) + (-c - 3c) \\ &= 10a - b - 4c.\end{aligned}$$

This solution can be written directly, as

$$(2a - 5b - c) + (8a + 4b - 3c) = 10a - b - 4c,$$

or it can be arranged so that like terms appear in columns, as

$$\begin{array}{r} 2a - 5b - c \\ 8a + 4b - 3c \\ \hline 10a - b - 4c \end{array}$$

in which the columns are added.

• PROBLEM 1-31

From the sum of $6x^2 + 4xy - 8y^2 - 11$ and $3x^2 - 4y^2 + 8 + 5xy$ subtract $xy - 10 - 5x^2 + 7y^2$.

Solution: First find the sum of $6x^2 + 4xy - 8y^2 - 11$ and $3x^2 - 4y^2 + 8 + 5xy$. Adding these two polynomials together:

$$(6x^2 + 4xy - 8y^2 - 11) + (3x^2 - 4y^2 + 8 + 5xy)$$

$$= 6x^2 + 4xy - 8y^2 - 11 + 3x^2 - 4y^2 + 8 + 5xy$$

Grouping like terms together,

$$(6x^2 + 4xy - 8y^2 - 11) + (3x^2 - 4y^2 + 8 + 5xy)$$

$$= (6x^2 + 3x^2) + (4xy + 5xy) + (-8y^2 - 4y^2) + (-11 + 8)$$

$$= 9x^2 + 9xy + (-12y^2) + (-3) = 9x^2 + 9xy - 12y^2 - 3 \qquad (1)$$

Now subtract $xy - 10 - 5x^2 + 7y^2$ from the resultant sum, which is the right side of equation (1), or $9x^2 + 9xy - 12y^2 - 3$. Then,

$$(9x^2 + 9xy - 12y^2 - 3) - (xy - 10 - 5x^2 + 7y^2) =$$

$$= 9x^2 + 9xy - 12y^2 - 3 - xy + 10 + 6x^2 - 7y^2 \qquad (2)$$

Grouping like terms together, equation (2) becomes:

$$(9x^2 + 9xy - 12y^2 - 3) - (xy - 10 - 5x^2 + 7y^2)$$

$$= (9x^2 + 5x^2) + (9xy - xy) + (-12y^2 - 7y^2) + (-3 + 10)$$

$$= 14x^2 + 8xy + (-19y^2) + 7$$

$$= 14x^2 + 8xy - 19y^2 + 7,$$

which is the final answer.

• PROBLEM 1-32

Simplify $3ax(ax^2 - 5bx + c)$.

Solution: Using the distributive property,

$$3ax(ax^2 - 5bx + c) = 3ax(ax^2) + 3ax(-5bx) + 3ax(c)$$
$$= 3a^2x^3 - 15abx^2 + 3acx \ .$$

• PROBLEM 1-33

Expand $(x + 5)(x - 4)$.

Solution: Distributing the second term:

$$(x + 5)(x - 4) = (x + 5)x + (x + 5)(-4) \qquad (1)$$

Distributing twice on the right side of equation (1):

$$(x + 5)(x - 4) = (x + 5)x + (x + 5)(-4)$$
$$= (x^2 + 5x) + (-4x - 20)$$
$$= x^2 + 5x - 4x - 20$$
$$= x^2 + x - 20.$$

• PROBLEM 1-34

Multiply $(2x + 3y)(4x - 5y)$.

Solution: Instead of writing one factor beneath the other, we shall use the following process to find the product mentally. Multiply the two first terms, $2x$ and $4x$, to obtain the first term in the product; also, multiply the two last terms, $3y$ and $-5y$, to obtain the last term in the product. Thus, we have $8x^2$ and $-15y^2$ as the first and last terms, respectively. We must now determine the middle term in the desired product. This is done by multiplying the two inner terms, $3y$ and $4x$, and the two outer terms, $2x$ and $-5y$, and adding these. This is shown below, and the middle term is $12xy - 10xy = 2xy$.

$$(2x + 3y)(4x - 5y) = 8x^2 + 2xy - 15y^2$$

12xy

-10xy

It is to be noted that the final result is written immediately with only one intermediate step: the two cross products, $12xy$ and $-10xy$, are kept in mind and added mentally to produce the middle term $2xy$.

• PROBLEM 1-35

Divide $(37 + 8x^3 - 4x)$ by $(2x + 3)$.

Solution: Arrange both polynomials in descending powers of the variable. The first polynomial becomes: $8x^3 - 4x + 37$. The second polynomial stays the same: $2x + 3$. The problem is: $2x + 3\overline{)8x^3 - 4x + 37}$. In the dividend, $8x^3 - 4x + 37$, all powers of x must be included. The only missing power of x is x^2. To include this power of x, a coefficient of 0 is used; that is, $0x^2$. This term, $0x^2$, can be added to the

dividendwithout changing the dividend because $0x^2 = 0$ (anything multiplied by 0 is 0).

Now to accomplish the division we proceed as follows: divide the first term of the divisor into the first term of the dividend. Multiply the quotient from this division by each term of the divisor and subtract the products of each term from the dividend. We then obtain a new dividend. Use this dividend, and again divide by the first term of the divisor, and repeat all steps again until we obtain a remainder which is of degree lower than that of the divisor or zero. Following this procedure we obtain :

$$
\begin{array}{r}
4x^2 - 6x + 7 \\
2x + 3\overline{\big)\,8x^3 + 0x^2 - 4x + 37} \\
\underline{8x^3 + 12x^2} \\
- 12x^2 - 4x + 37 \\
\underline{- 12x^2 - 18x} \\
14x + 37 \\
\underline{14x + 21} \\
16
\end{array}
$$

The degree of a polynomial is the highest power of the variable in the polynomial.

The degree of the divisor is 1. The number 16 can be written as $16x^0$ where $x^0 = 1$. Therefore, the number 16 has degree 0. When the degree of the divisor is greater than the degree of the dividend, we stop dividing.

Since the degree of the divisor in this problem is 1 and the degree of the dividend (16) is 0, the degree of the divisor is greater than the degree of the dividend. Therefore, dividing is stopped and the remainder is 16. Therefore, the quotient is $4x^2 - 6x + 7$ and the remainder is 16.

In order to verify this,multiply the quotient, $4x^2 - 6x + 7$, by the divisor, $2x + 3$, and then add 16. These two operations should total up to the dividend $8x^3 - 4x + 37$. Thus,

$$(4x^2 - 6x + 7)(2x + 3) + 16 =$$

$$8x^3 - 12x^2 + 14x + 12x^2 - 18x + 21 + 16 =$$

$$8x^3 - 4x + 37,$$

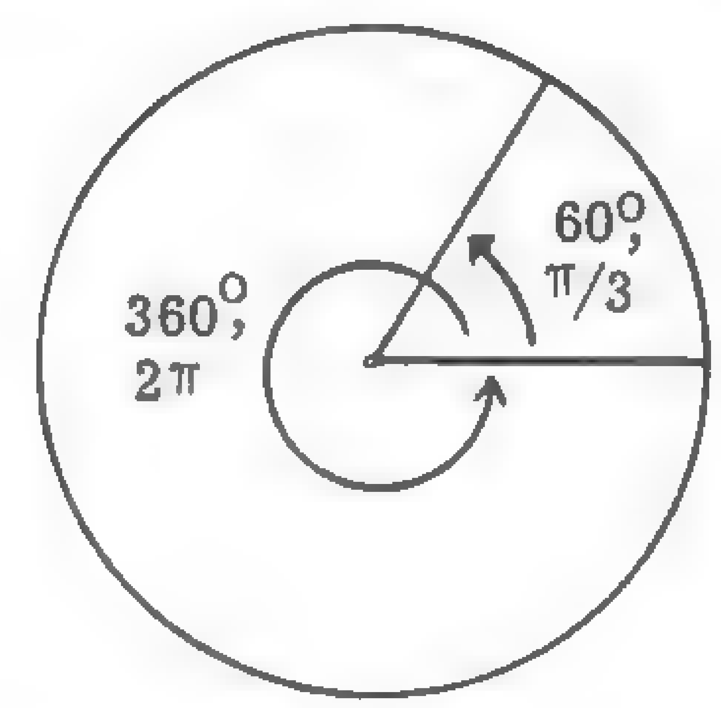
which is the desired result.

CHAPTER 2

BASIC TRIGONOMETRIC RELATIONS

• PROBLEM 2-1

Convert the measures 30°, 45°, 60°, 90°, 180° from degrees to radians.



Solution: There are two systems for measuring angles and arc-length, degree-measure and radian-measure. Degree measure is most useful when working with many angles, since 360, the number of degrees in a full circle, is easily divisible into fractions without remainders ($360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$). On the other hand, when relating arc lengths to other distance measures, radian measure is the preferred system: Arc length = (angle in radians)(radius).

In our example, the most direct means of conversion will be to express the degree measure as a fraction of 360°, and multiply by 2π, i.e., the conversion factor in the transformation. This conversion factor is actually an adjustment of units and equals $2\pi/360$ radians per degree.

To understand how the conversion factor is derived, let D be the measure of the angle in degrees and R be the measure of the angle in radians. Note that the given angle

is always a certain fraction of the circular angle regardless of the units used in measuring the angle. A circular angle has $360°$ or 2π radians.

For example, a vertex angle of an equilateral triangle must always be one sixth of a circular angle: $\frac{60°}{360°} = \frac{1}{6}$ and $\frac{\pi/3}{2\pi} = \frac{1}{6}$. Thus, it must be always true that:

$$\text{(I)} \qquad \frac{D}{360} = \frac{R}{2\pi}$$

Solving for R, we obtain

$$\text{(II)} \qquad R = \frac{2\pi}{360} D$$

Thus given any degree measure of an angle D, multiplying by $\frac{2\pi}{360}$ yields the radian measure. Therefore,

$$30° \cdot \frac{2\pi}{360} = \frac{1}{12} \cdot 2\pi = \frac{\pi}{6} \text{ radians.}$$

$$45° \cdot \frac{2\pi}{360} = \frac{1}{8} \cdot 2\pi = \frac{\pi}{4} \text{ radians.}$$

$$60° \cdot \frac{2\pi}{360} = \frac{1}{6} \cdot 2\pi = \frac{\pi}{3} \text{ radians.}$$

$$90° \cdot \frac{2\pi}{360} = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2} \text{ radians.}$$

$$180° \cdot \frac{2\pi}{360} = \frac{1}{2} \cdot 2\pi = \pi \text{ radians.}$$

• PROBLEM 2-2

If $\tan\theta = 3.8436$, find θ.

Solution: Looking through a table of trigonometric functions under the vertical column marked tan, it is found that the angle $\theta = 75°35'$ corresponds to the number 3.8436.

• PROBLEM 2-3

Complete the following table:

	1	2	3	4	5	6	7	8	9
Width of θ in radians	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$		$\frac{1}{2}\pi$	$\frac{2}{3}\pi$			π
Width of θ in degrees	$0°$			$60°$			$135°$	$150°$	

Solution: If an angle θ is A degrees wide and also t radians wide, then the numbers A and t are related by the equation:

$$\frac{A}{180°} = \frac{t}{\pi} \qquad (1)$$

Thus, equation (1) can be used to complete the table. For column (2):

$$\frac{A}{180°} = \frac{1/6\,\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180°} = 1/6$$

Multiplying both sides by $180°$,

$$\cancel{180°}\left(\frac{A}{\cancel{180°}}\right) = 180°(1/6)$$

$$A = 30°$$

For column (3):

$$\frac{A}{180°} = \frac{\frac{1}{4}\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180°} = \frac{1}{4}$$

Multiplying both sides by $180°$,

$$\cancel{180°}\left(\frac{A}{\cancel{180°}}\right) = 180°(\tfrac{1}{4})$$

$$A = 45°$$

For column (4):

$$\frac{60°}{180°} = \frac{t}{\pi}$$

$$\frac{1}{3} = \frac{t}{\pi}$$

Multiplying both sides by π,

$$\pi(1/3) = \cancel{\pi}(t/\cancel{\pi})$$

$$1/3\,\pi = t$$

For column (5):

$$\frac{A}{180°} = \frac{\frac{1}{2}\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180°} = \frac{1}{2}$$

Multiplying both sides by $180°$,

$$\cancel{180°}\left(\frac{A}{180°}\right) = 180°(\tfrac{1}{2})$$

$$A = 90°$$

For column (6):

$$\frac{A}{180^\circ} = \frac{2/3\,\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180^\circ} = 2/3$$

Multiplying both sides by 180°,

$$\cancel{180^\circ}\left(\frac{A}{\cancel{180^\circ}}\right) = 180^\circ(2/3)$$

$$A = 120^\circ$$

For column (7):

$$\frac{135^\circ}{180^\circ} = \frac{t}{\pi}$$

$$\frac{27}{36} = \frac{t}{\pi}$$

$$\frac{3}{4} = \frac{t}{\pi}$$

Multiplying both sides by π,

$$\pi(3/4) = \cancel{\pi}(t/\cancel{\pi})$$

$$3/4\ \pi = t$$

For column (8):

$$\frac{150^\circ}{180^\circ} = \frac{t}{\pi}$$

$$\frac{50}{60} = \frac{t}{\pi}$$

$$\frac{5}{6} = \frac{t}{\pi}$$

Multiplying both sides by π, $\pi(\frac{5}{6}) = \cancel{\pi}(\frac{t}{\cancel{\pi}})$.

$$\frac{5}{6}\pi = t$$

For column (9):

$$\frac{A}{180^\circ} = \frac{\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180^\circ} = 1$$

Multiplying both sides by 180°,

$$\cancel{180^\circ}\left(\frac{A}{\cancel{180^\circ}}\right) = 180^\circ(1)$$

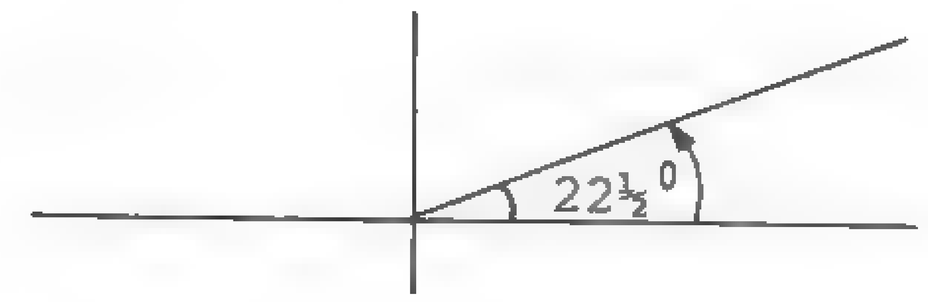
$$A = 180^\circ$$

All of the computed values are now put into the table as follows:

	1	2	3	4	5	6	7	8	9
Width of θ in radians	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
Width of θ in degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°

• PROBLEM 2-4

Find $\sin 22\frac{1}{2}^\circ$, $\cos 22\frac{1}{2}^\circ$, $\tan 22\frac{1}{2}^\circ$, and $\cot 22\frac{1}{2}^\circ$.



Solution: The angle $22\frac{1}{2}^\circ$, or $22^\circ\ 30'$, is a first quadrant angle (see the figure). Therefore, all the required functions (sine, cosine, tangent and cotangent) are positive. Hence, from a table of trigonometric functions, $\sin 22\frac{1}{2}^\circ = \sin 22^\circ 30' = 0.3827$, $\cos 22\frac{1}{2}^\circ = \cos 22^\circ 30' = 0.9239$, $\tan 22\frac{1}{2}^\circ = \tan 22^\circ 30' = 0.4142$, and $\cot 22\frac{1}{2}^\circ = \cot 22^\circ 30' = 2.4142$.

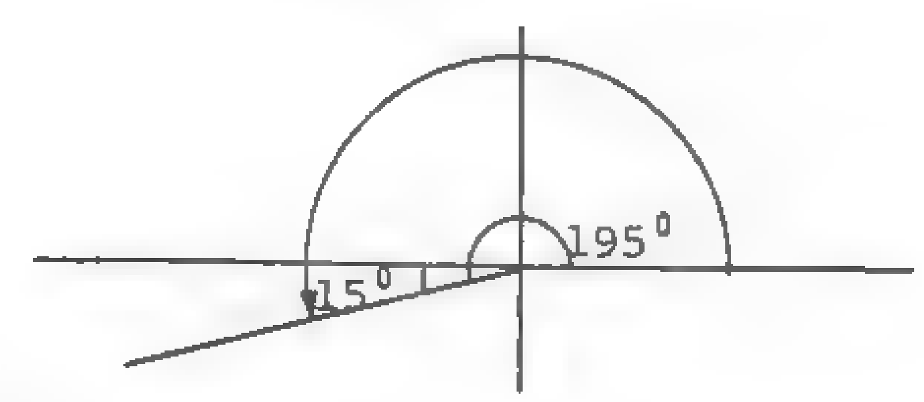
• PROBLEM 2-5

Find $\tan 635^\circ 19'$.

Solution: The reference angle of $635^\circ 19'$ is $2(360^\circ 00') - 635^\circ 19' = 84^\circ 41'$. Therefore, the $\tan 635^\circ 19' = \tan 84^\circ 41' = 10.746$. (This value may be found from a table of trigonometric functions.) However, the angle $635^\circ 19'$ is a fourth quadrant angle and the tangent function is negative in the fourth quadrant. Hence, $\tan 635^\circ 19' = -\tan 84^\circ 41' = -10.746$.

• PROBLEM 2-6

Find $\sin 195^\circ$, $\cos 195^\circ$, $\tan 195^\circ$, $\cot 195^\circ$.



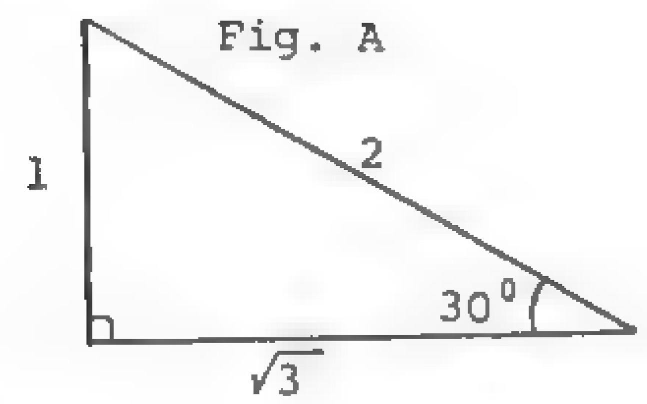
Solution: The reference angle for 195° is 15°. Also, 195° is a third quadrant angle (see Figure)

In the third quadrant, the sine and cosine functions are negative, but the tangent and the cotangent functions are positive. Therefore, $\sin 195^\circ = \sin 15^\circ = -0.2588$, $\cos 195^\circ = \cos 15^\circ = -0.9659$, $\tan 195^\circ = \tan 15^\circ =$

0.2679, and cot 195° = cot 15° = 3.7321. (Note that the values obtained for the trigonometric functions were found in a table of trigonometric functions.)

• PROBLEM 2-7

Find (a) tan 30° (b) tan 90°



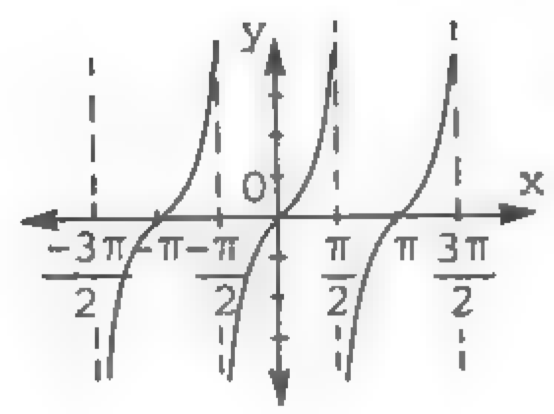


Fig. B

Solution: a) Recall $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, by definition; hence

$$\tan 30° = \frac{\sin 30°}{\cos 30°}$$

Looking at a 30-60 right triangle we find the values of sin 30° and cos 30°:

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}}\ ;\ \sin 30° = \tfrac{1}{2}$$

$$\cos = \frac{\text{adjacent side}}{\text{hypotenuse}}\ ;\ \cos 30° = \frac{\sqrt{3}}{2}$$

hence, $$\frac{\sin 30°}{\cos 30°} = \frac{\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}$$

Multiplying numerator and denominator by 2:

$$= \frac{2(\tfrac{1}{2})}{2\left(\tfrac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}} \quad \text{therefore } \tan 30° = \frac{1}{\sqrt{3}}$$

b) To determine tan 90° we can use the same method as in part a:

$$\tan 90° = \frac{\sin 90°}{\cos 90°}\text{, by definition of tangent.}$$

$$= \frac{1}{0}\ .$$

However, since the quotient $\frac{1}{0}$ is undefined, we must therefore conclude that the tangent of 90° does not exist.

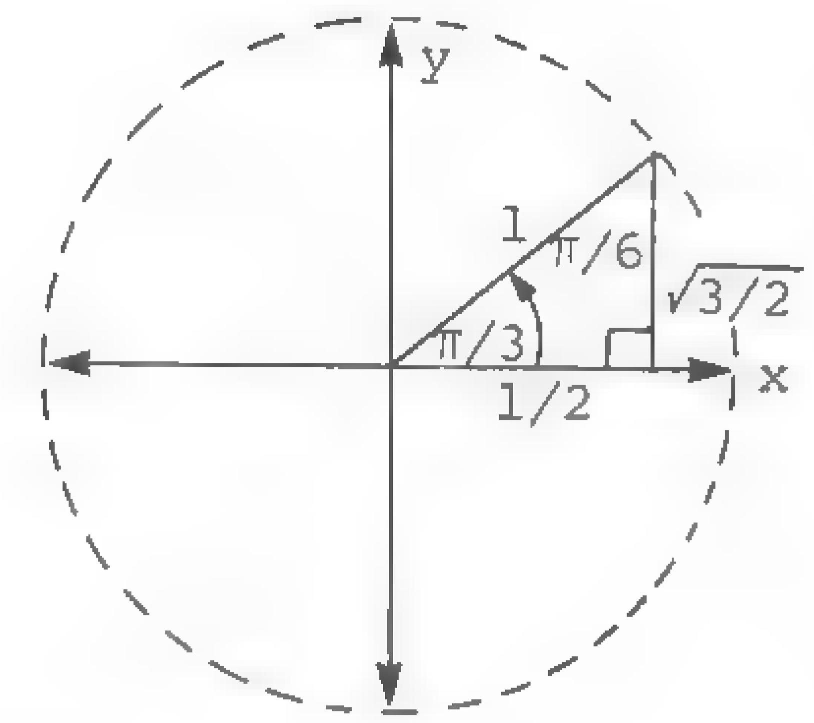
We may also observe the graph of the tangent function to determine tan 90°.

$$\tan 90° = \tan\frac{90 \cdot \pi}{180}\ \text{radians} = \tan\frac{\pi}{2}$$

We observe that the tangent function does not exist on the line $\frac{\pi}{2}$ (the graph of the tangent function is asymptodic to the line $x = \frac{\pi}{2}$, but never touches it).

• PROBLEM 2-8

Calculate the values of the six trigonometric functions at the point $\frac{1}{3}\pi$.



Solution: To find the trigonometric point $P\left(\frac{1}{3}\pi\right)$, proceed around the unit circle in a counterclockwise direction, since $\frac{\pi}{3}$ is a positive angle. Recall that $\sin 60°$ i.e., $\sin(\pi/3) = \sqrt{3}/2$. Now, using the Pythagorean theorem and the fact that the hypotenuse is unity because it is a unit circle we can compute the third side, which we find to be 1/2 (see figure). Therefore, the coordinates of the trigonometric point $P\left(\frac{1}{3}\pi\right)$ are $(1/2,\ 1/2\sqrt{3})$. Hence, we apply the following equations:

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\text{adjacent side}}{\text{opposite side}}$$

Thus,

$$\cos\tfrac{1}{3}\pi = \tfrac{1}{2}, \qquad \sec\tfrac{1}{3}\pi = 2,$$

$$\sin\tfrac{1}{3}\pi = \tfrac{1}{2}\sqrt{3}, \qquad \csc\tfrac{1}{3}\pi = 2/\sqrt{3} = 2/\sqrt{3}\cdot\sqrt{3}/\sqrt{3} = 2/3\,\sqrt{3}$$

$$\tan\tfrac{1}{3}\pi = \sqrt{3}, \qquad \cot\tfrac{1}{3}\pi = 1/\sqrt{3} = 1/\sqrt{3}\cdot\sqrt{3}/\sqrt{3} = 1/3\,\sqrt{3}$$

• **PROBLEM 2-9**

What primary angle is coterminal with the angle of -743°?

Solution: $$-743^\circ = \alpha - n \cdot 360^\circ \qquad (1)$$

$$-743^\circ = \alpha - 3 \cdot 360^\circ = \alpha - 1080^\circ \qquad (2)$$

Multiply both sides of equation (2) by -1.

$$-1(-743^\circ) = -1(\alpha - 1080^\circ)$$

$$743^\circ = -\alpha + 1080^\circ$$

$$743^\circ = 1080^\circ - \alpha \qquad (3)$$

Note that the positive integer value chosen for n results in an angle (in equation (3)) which is larger than but closest to the angle of 743°. Also,

$$0^\circ \leq \alpha \leq 360^\circ.$$

From equation (3),

$$\alpha = 1080^\circ - 743^\circ = 337^\circ.$$

• **PROBLEM 2-10**

What primary angle is coterminal with the angle of 1243°?

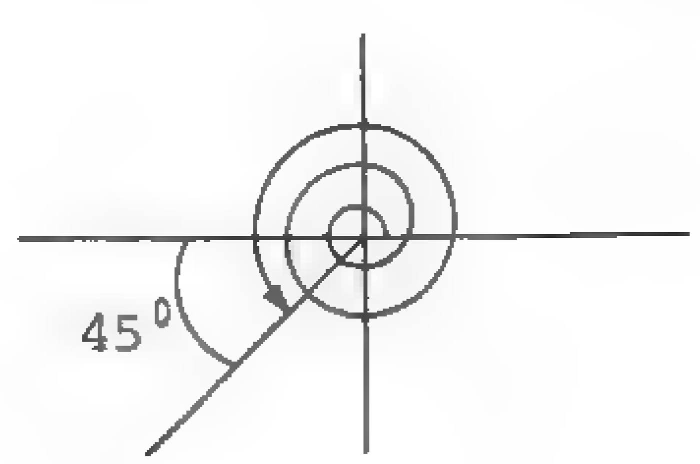
Solution: Using the formula, $\beta = \alpha + n \cdot 360^\circ$, obtain $1243^\circ = \alpha + n \cdot 360^\circ$. In this formula choose the largest positive integer for n which, when multiplied by 360, is closest to but smaller than the given angle. Also, α is an angle between 0° and 360°, that is

$$0^\circ \leq \alpha \leq 360^\circ.$$

Since $3 \cdot 360^\circ = 1080^\circ$ and $4 \cdot 360^\circ = 1440^\circ$, $n = 3$, and $1243^\circ = \alpha + 1080^\circ$ or $\alpha = 1243^\circ - 1080^\circ = 163^\circ$. Thus, the angles 1243° and 163° are coterminal.

• **PROBLEM 2-11**

What primary angle is coterminal with the angle of $5\frac{1}{4}\pi$ radians?

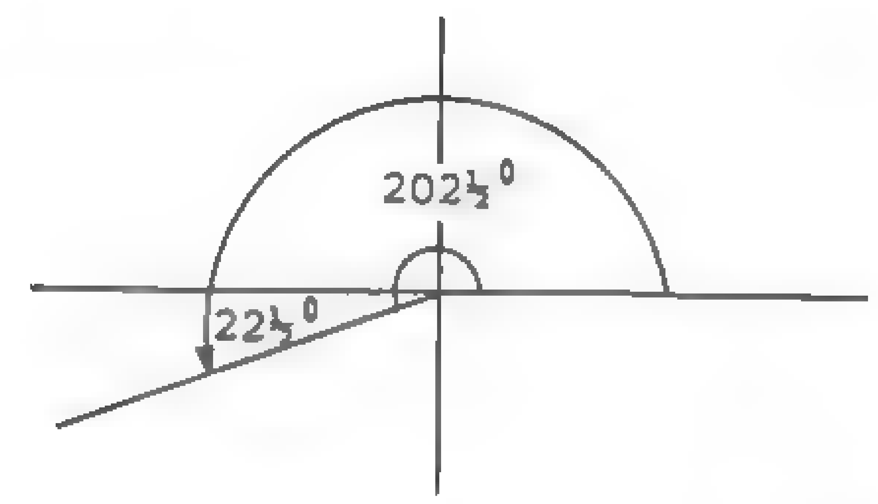


Solution: The figure illustrates the angle of $5\frac{1}{4}\pi$ radians.

Note that the angle of $5\frac{1}{4}\pi$ radians has a reference angle of 45°. However, we seek a primary angle (an angle between 0° and 360°) which is coterminal with $5\frac{1}{4}\pi$; that is, which has the same terminal side as an angle of $5\frac{1}{4}\pi$ radians. Also, since a primary angle is a positive angle, its initial side is the positive x-axis and the angle revolves in the counter-clockwise direction. Therefore, a primary angle with the same terminal side as an angle of $5\frac{1}{4}\pi$ radians is $(180^\circ + 45^\circ) = 225^\circ = (\pi + \pi/4)$ radians $= 5/4\ \pi$ radians.

• **PROBLEM 2-12**

Find $\sin 202\frac{1}{2}^\circ$ and $\tan 202\frac{1}{2}^\circ$.

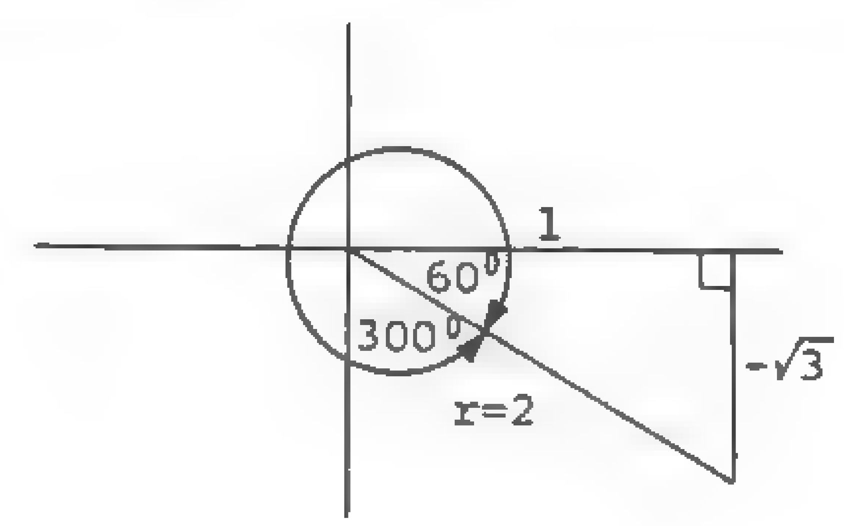


Solution: $202\frac{1}{2}^\circ$ is a third quadrant angle. Thus its reference angle is $22\frac{1}{2}^\circ$ $(202\frac{1}{2}^\circ - 180^\circ = 22\frac{1}{2}^\circ)$. See the figure.

In the third quadrant, the sine function is negative while the tangent function is positive. Therefore, using a table of trigonometric functions, $\sin 202\frac{1}{2}^\circ = -\sin 22\frac{1}{2}^\circ = -\sin 22^\circ 30' = -0.3827$ and $\tan 202\frac{1}{2}^\circ = \tan 22\frac{1}{2}^\circ = \tan 22^\circ 30' = 0.4142$.

• **PROBLEM 2-13**

Find the values of the trigonometric functions of an angle of 300°.



Solution: An angle of 300° is a fourth quadrant angle and its reference angle is an angle of 60°. In the fourth quad-

rant the sine, tangent, cotangent and cosecant functions are negative. This yields

$$\sin 300° = \sin 60° = -\frac{\sqrt{3}}{2},$$

$$\cos 300° = \cos 60° = \frac{1}{2},$$

$$\tan 300° = \tan 60° = -\sqrt{3},$$

$$\cot 300° = \cot 60° = -\frac{\sqrt{3}}{3},$$

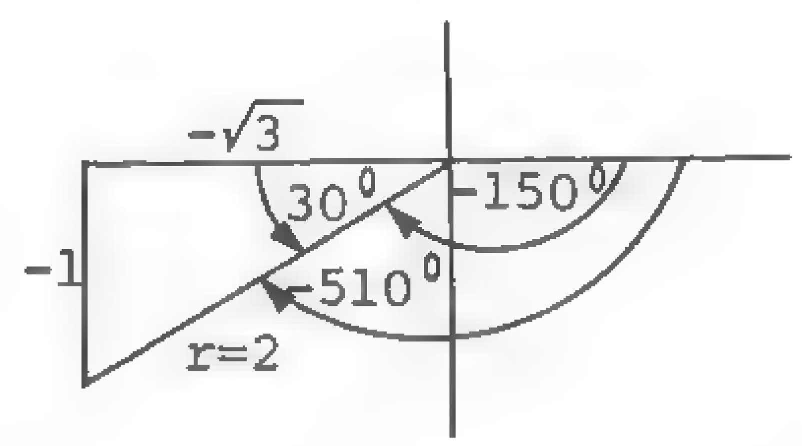
$$\sec 300° = \sec 60° = \frac{1}{\cos 60°} = \frac{1}{\frac{1}{2}} = 2, \text{ and}$$

$$\csc 300° = \csc 60° = \frac{1}{\sin 60°} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

$$= -\frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = -\frac{2\sqrt{3}}{3}.$$

• PROBLEM 2-14

Find the values of the trigonometric functions of an angle of - 510°.



Solution: We see that angles of - 510° and - 150° are coterminal angles having the same values for the trigonometric functions. The reference angle of an angle of - 150° is an angle of 30° and - 150° is a third quadrant angle. In the third quadrant, the tangent and cotangent functions are positive while the other four functions are negative. This yields

$$\sin(-510°) = \sin 30° = -\frac{1}{2},$$

$$\cos(-510°) = \cos 30° = -\frac{\sqrt{3}}{2},$$

$$\tan(-510°) = \tan 30° = \frac{\sqrt{3}}{3},$$

$$\cot(-510°) = \cot 30° = \frac{1}{\tan 30°} = \frac{1}{\sqrt{3}/3} = \frac{3}{\sqrt{3}}$$

$$= \frac{3\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3},$$

$$\sec(-510°) = \sec 30° = \frac{1}{\cos 30°} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

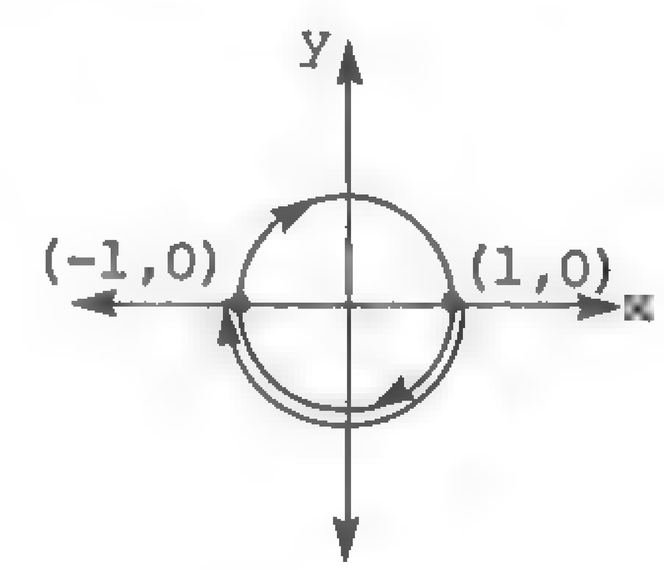
$$= -\frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = -\frac{2\sqrt{3}}{3},$$

$$\csc(-510°) = \csc 30° = \frac{1}{\sin 30°} = \frac{1}{-\frac{1}{2}} = -2.$$

• PROBLEM 2-15

Calculate the values of the six trigonometric functions at the point -3π.

Solution: To locate the trigonometric point $P(-3\pi)$ we proceed 3π units around the unit circle in a clockwise direction, and we find that the coordinates of $P(-3\pi)$ are $(-1,0)$. See the accompanying figure. On the figure, proceeding from the point $(1,0)$ to $(-1,0)$ is π units; back to $(1,0)$ is π more units (or 2π); and returning to $(-1,0)$ is another π units (or 3π). Hence,



$$\cos(-3\pi) = -1, \qquad \tan(-3\pi) = \frac{\sin(-3\pi)}{\cos(-3\pi)} = 0,$$

$$\sin(-3\pi) = 0, \qquad \sec(-3\pi) = \frac{1}{\cos(-3\pi)} = -1.$$

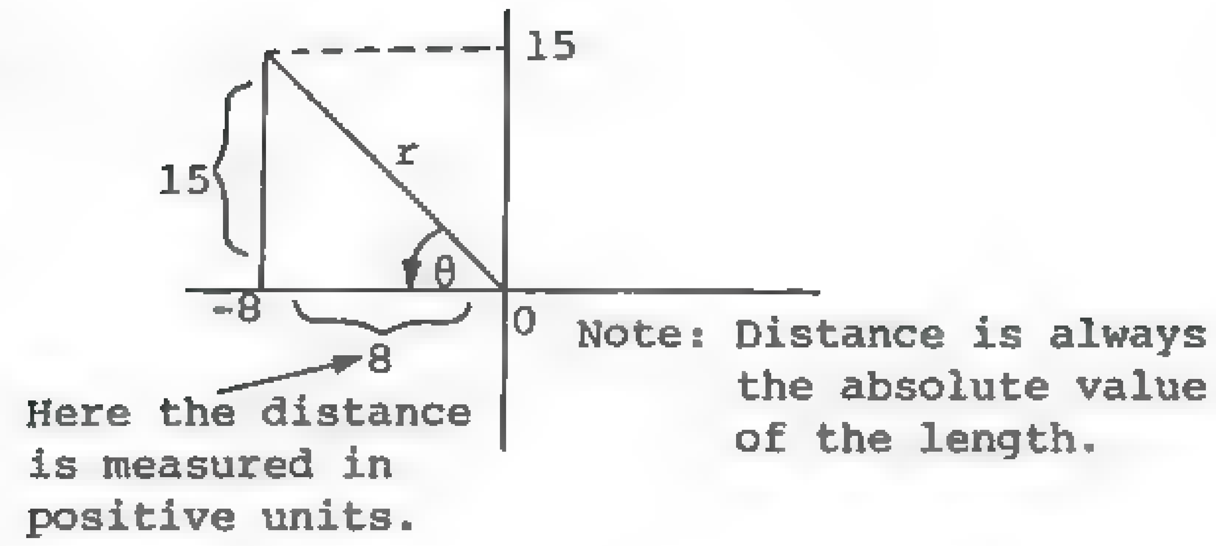
Since the Y-coordinate of $P(-3\pi)$ is 0, the symbols $\cot(-3\pi)$ and $\csc(-3\pi)$ are not defined.

$$\cot(-3\pi) = \frac{1}{\tan(-3\pi)} = \frac{1}{0} = \text{undefined}$$

$$\csc(-3\pi) = \frac{1}{\sin(-3\pi)} = \frac{1}{0} = \text{undefined}.$$

• PROBLEM 2-16

Given that θ is an angle in the second quadrant and that $\tan\theta = -\frac{15}{8}$, find the other functions of θ.



Solution: The equations which relate the trigonometric functions of an angle to the sides of a right triangle will be used to solve this problem. Therefore,

$$\tan\theta = \frac{y}{x} = -\frac{15}{8}.$$

Also, θ is a second quadrant angle where $y > 0$ and $x < 0$. Therefore, $y = 15$ and $x = -8$. Thus,

$$x^2 + y^2 = (-8)^2 + (15)^2 = 64 + 225 = 289 = r^2.$$

Therefore, $r = \sqrt{289} = 17$. Then

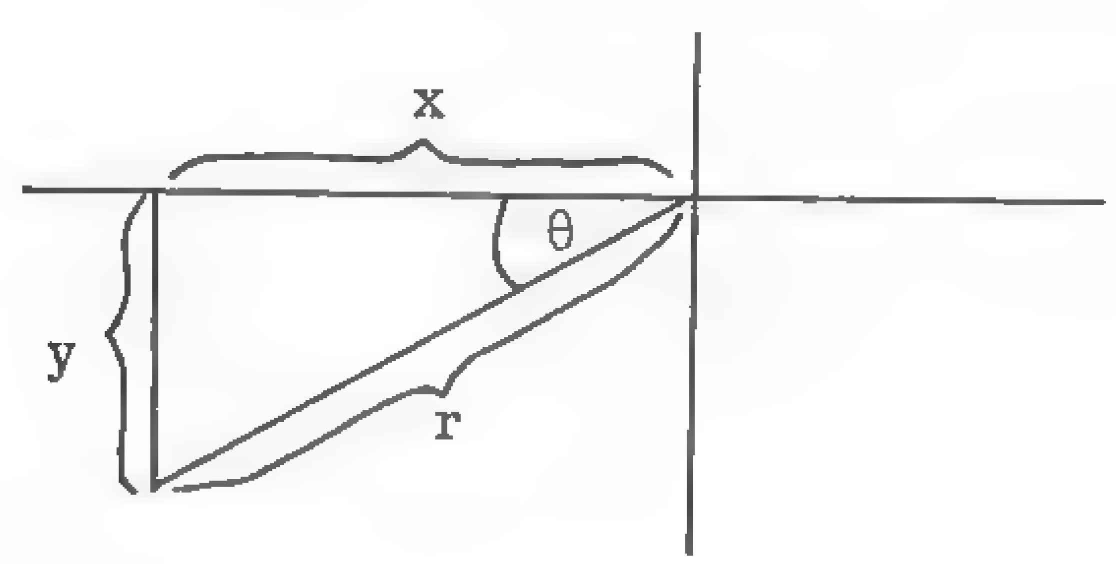
$$\sin\theta = \frac{15}{17}, \qquad \cos\theta = \frac{-8}{17} = -\frac{8}{17},$$

$$\tan\theta = -\frac{15}{8}, \qquad \cot\theta = \frac{-8}{15} = -\frac{8}{15},$$

$$\sec\theta = \frac{17}{-8} = -\frac{17}{8}, \text{ and } \qquad \csc\theta = \frac{17}{15}.$$

• PROBLEM 2-17

Given that $\tan\theta = 2$ and $\cos\theta$ is negative, find the other functions of θ.



Solution: Since cos θ is negative, θ must be a second or third quadrant angle. In the second quadrant, the tangent function is negative. Hence, θ must be a third quadrant angle.

In the figure, the trigonometric functions have the following values:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x},$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x/r} = \frac{r}{x}, \qquad \text{and}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y/r} = \frac{r}{y}.$$

Also, from the figure, $r^2 = x^2 + y^2$ (from the Pythagorean Theorem), or $r = \sqrt{x^2 + y^2}$. Therefore, in this problem,

$$\tan\theta = 2 = \frac{-2}{-1} = \frac{y}{x}.$$

Hence, $y = -2$ and $x = -1$. Also, in this problem,

$$r^2 = x^2 + y^2 = (-1)^2 + (-2)^2 = 1 + 4$$

or $r^2 = 5$ or $r = \sqrt{5}$. Therefore,

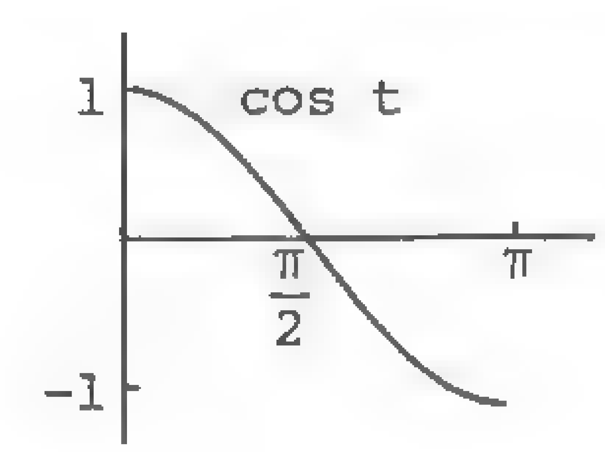
$$\sin\theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}, \qquad \cos\theta = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5},$$

$$\tan\theta = 2, \qquad \cot\theta = \frac{-1}{-2} = \frac{1}{2},$$

$$\sec\theta = \frac{\sqrt{5}}{-1} = -\sqrt{5}, \text{ and } \qquad \csc\theta = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}.$$

• PROBLEM 2-18

If $0 < t < \frac{1}{2}\pi$, and $\tan t = \frac{1}{2}\sqrt{5}$, find cos t.



Solution: One way to find cos t is to express $\frac{1}{2}\sqrt{5}$ as a decimal, use a table of trigonometric functions to solve the equation $\tan t = \frac{1}{2}\sqrt{5}$ for t, and then again use the table to find cos t. But it is much easier to use some

trigonometric identities to solve this problem.

Since we are given that $\tan t = \frac{1}{2}\sqrt{5} = \frac{\sqrt{5}}{2}$, we can use the identity $\sec^2 t = 1 + \tan^2 t$ to find $\sec^2 t$. Substituting, we obtain:

$$\sec^2 t = 1 + (\tan t)^2$$

$$\sec^2 t = 1 + \left(\frac{\sqrt{5}}{2}\right)^2, \text{ or}$$

$$\sec^2 t = 1 + \frac{5}{4} = \frac{4}{4} + \frac{5}{4} = \frac{9}{4}$$

Thus, since $\cos t = \frac{1}{\sec t}$, then

$$\cos^2 t = 1/\sec^2 t = \frac{1}{\frac{9}{4}} = \frac{4}{9}; \text{ therefore}$$

$$\cos t = \sqrt{\frac{4}{9}} = \pm \frac{2}{3}$$

Since $0 < t < \frac{1}{2}\pi$, we know that $\cos t > 0$ (see graph) and hence we reject the negative value, $-\frac{2}{3}$, and it follows from the last equation that $\cos t = \frac{2}{3}$.

• PROBLEM 2-19

Find $\cos 37^\circ 12'$.

Solution: See a table of natural trigonometric functions, which is constructed in terms of multiples of ten seconds. The cosine of $37^\circ 12'$ lies between $37^\circ 10'$ and $37^\circ 20'$. Therefore, we must interpolate. The cosine decreases as the angle increases, so we form our proportion as follows, where

x = the cosine of the angle $37^\circ 12'$

d = the difference between the cos $37^\circ 10'$ and cos $37^\circ 12'$:

$$10' \left[\begin{array}{l} \cos 37^\circ 10' = 0.7969 \\ 2' \left[\cos 37^\circ 12' = x \right] d \\ \cos 37^\circ 20' = 0.7951 \end{array} \right] -0.0018$$

$$\frac{2}{10} = \frac{d}{-0.0018}$$

Cross multiply to obtain:

$$10d = 2(-0.0018)$$

$$d = .2(-0.0018)$$
$$= -0.00036$$
$$d \approx -0.0004$$

Thus,

$$x = 0.7969 - 0.0004$$
$$= 0.7965$$

Since the cosine is positive in the first quadrant,

$$\cos 37^\circ 12' = 0.7965$$

Remember that results obtained by interpolation are approximations. You should not use an answer that is more accurate than the original data, in this case, four significant digits.

• PROBLEM 2-20

Find the value of tan 38°46' by use of interpolation.

Solution: Since 38°46' is between 38°40' and 38°50', we assume that tan 38°36' is between tan 38°40' and tan 38°50'. In fact, since 38°46' is six-tenths of the way from 38°40' toward 38°50', we assume that tan 38°46' is six-tenths of the way from tan 38°40' = .8002 toward tan 38°50' = .8050. Using these assumptions we perform the following interpolation:

$$10' \left[\begin{array}{l} 6' \left[\begin{array}{l} \tan 38^\circ 40' = .8002 \\ \tan 38^\circ 46' = \ ? \end{array} \right] c \\ \tan 38^\circ 50 = .8050 \end{array} \right] .0048$$

Set up the proportion $\frac{c}{.0048} = \frac{6}{10}$

$$10c = 6(.0048)$$

$$c = \frac{6}{10}(.0048) = .0029$$

$$\tan 38^\circ 46' = .8002 + .0029 = .8031$$

Therefore, c was added because $\tan\theta$ increases from $\theta = 38^\circ 40'$ to $\theta = 38^\circ 50'$.

• PROBLEM 2-21

Find tan 63°19.27'.

Solution: The value of tan 63°19.27' can be found by interpolating the values of tan 63°19' and tan 63°20'.

		Degree	Value of function		
1	.27	63°19'	1.9897	d	0.0015
		63° 19.27'	x		
		63° 20'	1.9912		

Now, set up the following proportion:

$$\frac{d}{0.0015} = \frac{.27}{1} = .27$$

$$d = (.27)(0.0015)$$

$$d = 0.0004.$$

Therefore, $x = \tan 63°19.27' = \tan 63°19' + d$

$$= 1.9897 + 0.0004$$

$$\tan 63°19.27' = 1.9901.$$

• PROBLEM 2-22

Find θ if sin θ = .6212, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Solution: Since 0.6212 is not found in the sine table, we proceed by finding the two numbers closest to .6212, one greater and the other less than it, and interpolating.

10'	c	sin 38°20'	= .6302	.0010	.0023
		sin θ	= .6212		
		sin 38°30'	= .6225		

We set up the proportion $\frac{c}{10'} = \frac{.0010}{.0023}$.

$$c = \frac{.0010}{.0023}(10') = \frac{10}{23}(10')$$

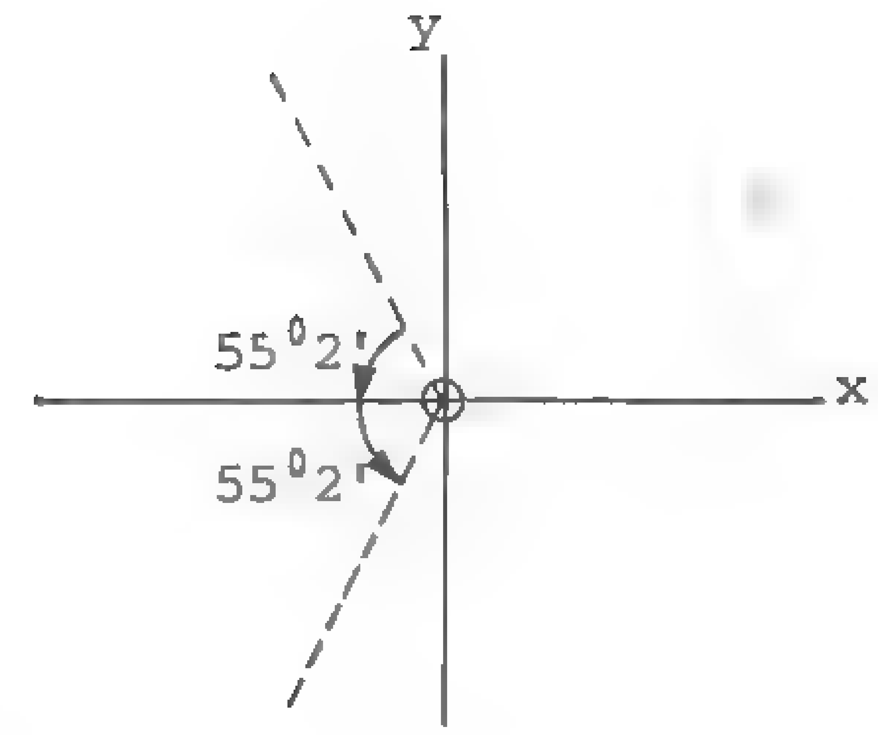
θ = 4' to the nearest minute.

Thus, θ = 38°20' + 4'

= 38°24'.

• PROBLEM 2-23

Find θ if cos θ = −0.5731 and 0 ≤ θ ≤ 360°.



Solution: Since 0.5731 is not found in the cosine table, we proceed by finding the two numbers closest to .5731, one greater and the other less than it, and interpolating. Notice that the cosine function decreases as the angle increases.

10'	d	cos 55°0'	= 0.5736	0.0005	0.0024
		cos x	= 0.5731		
		cos 55°10'	= 0.5712		

Setting up the proportion, $\frac{d}{10} = \frac{0.0005}{0.0024} = \frac{5}{24}$. Cross multiplying, 24d = 50

d = 2' to the nearest minute.

Thus, x = 55°0' + 2' = 55°2'. Since we are given a negative cosine, and cosine is negative in the second and third quadrants, we know that our reference angle 55°2' appears in the second or third quadrant (see diagram). Hence

= 180° − 55°2' = 124°58', or

= 180° + 55°2' = 235°2'.

• PROBLEM 2-24

Find a solution of the equation cos t = .6241.

Solution: We see that .6241 is not found in a trigonometric table; therefore, interpolation is necessary. We choose the two entries in the trigonometric table which .6241 lies between, and arrange the numbers as follows:

cos .89 = .6294,

cos t = .6241,

cos .90 = .6216.

We obtain,

$$.01\left[\begin{array}{l} x\left[\begin{array}{l}\cos .89 = .6294 \\ \cos t = .6241\end{array}\right]53 \\ \cos .90 = .6216\end{array}\right]78$$

by subtracting .89 from .90 (.90 - .89 = .01) and calling the difference between .89 and t, x. Also, .6294 - 6241 = .0053, and .6294 - .6216 = .0078. Since both .0053 and .0078 are four decimal places we can rewrite them as 53 and 78, without changing the value of the following proportion. Now,

$$\frac{x}{.01} = \frac{53}{78}$$

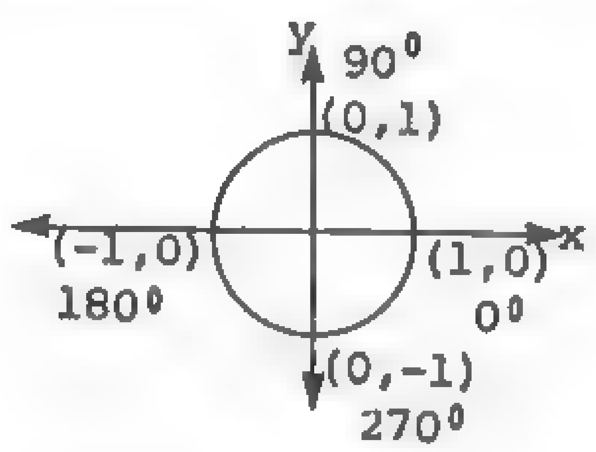
$$78x = 53(.01) = .53$$

$$x = \frac{.53}{78} \simeq .007$$

The cost lies between .89 and .90; thus, t = .89 +.007 = .897.

• PROBLEM 2-25

Find log sin 36°41′.



Solution: From a table of logarithms of trigonometric functions we find log sin 36°40′ and log sin 36°50′. Then, by the process of interpolation we find: log sin 36°41′ = 9.77626 - 10. Since all values of the sine function for acute angles are in the range of $0 < \sin x \leq 1$, the characteristic is negative. (Recall that for a number less than one, the characteristic is negative.) The range of sine can be seen by inspecting the accompanying figure. Sine is given by the y coordinate; cos is given by the x coordinate. Observe that y value varies from 0 to 1, as the angle varies from 0° to 90°.

• PROBLEM 2-26

Find log cos 49°13.6′.

Solution: First consult a table of logarithms of trigonometric functions.

Notice that 49°13.6′ lies between 49°10′ and 49°20′,

so that the log of 49°13.6′ will occur between the logs of 49°10′ and 49°20′ and can be determined by interpolation.

	angle	log cosine	
10′	49°20′	9.8140-10	.0015
3.6′	49°13.6′	x	
	49°10′	9.8155-10	

Now set up the proportion

$$\frac{3.6}{10} = \frac{x}{.0015} \quad \text{or} \quad 10x = .00540$$

$$x = .000540$$

Since cos decreases on the interval $0 \leq \theta < \pi/2$, subtract x from the log cosine of 49°10′.

$$\begin{array}{r} 9.8155 \\ -\ \ .000540 \\ \hline 9.814960 \end{array}$$

Thus, log cosine 49°13.6′ is 9.81496-10.

• PROBLEM 2-27

Reduce the expression $\dfrac{\tan x - \cot x}{\tan x + \cot x}$ to one involving only sin x.

Solution: Since, by definition, $\tan x = \dfrac{\sin x}{\cos x}$ and

$$\cot x = \frac{1}{\tan x} = \frac{1}{\sin x/\cos x} = \frac{\cos x}{\sin x},$$

$$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}$$

$$= \frac{\dfrac{\sin x(\sin x)}{\sin x(\cos x)} - \dfrac{\cos x(\cos x)}{\cos x(\sin x)}}{\dfrac{\sin x(\sin x)}{\sin x(\cos x)} + \dfrac{\cos x(\cos x)}{\cos x(\sin x)}}$$

$$= \frac{\dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x}}{\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$$

$$= \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} \times \frac{\sin x \cos x}{\sin^2 x + \cos^2 x}$$

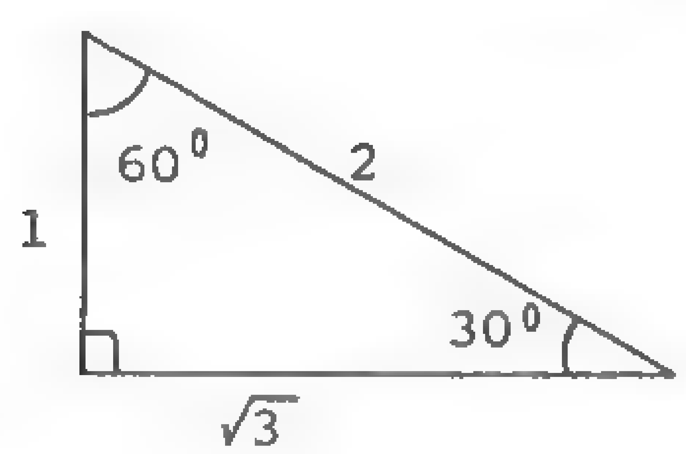
$$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x}$$

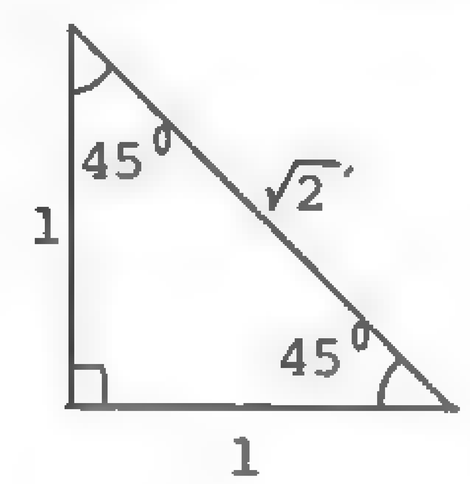
Since $\sin^2 x + \cos^2 x = 1$ or $\cos^2 x = 1 - \sin^2 x$,

$$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{1}$$
$$= \sin^2 x - \cos^2 x$$
$$= \sin^2 x - (1 - \sin^2 x)$$
$$= \sin^2 x - 1 + \sin^2 x$$
$$= 2 \sin^2 x - 1.$$

• PROBLEM 2-28

Find $\sin 105^\circ$ without the use of a trig. table.





Solution: We note that $105^\circ = 60^\circ + 45^\circ$ and find the sine of the sum of two angles.

$$\sin 105^\circ = \sin(60^\circ + 45^\circ)$$

Using the formula for the sine of the sum of two numbers,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,\ \sin(60^\circ + 45^\circ)$$
$$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ .$$

Now we must find the values of $\sin 60^\circ$, $\cos 45^\circ$, $\cos 60^\circ$, and $\sin 45^\circ$. Observing a 30-60 and 45-45 right triangle we note:

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}} ; \text{ thus, } \sin 60^\circ = \frac{\sqrt{3}}{2}$$
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos = \frac{\text{adjacent}}{\text{hypotenuse}} ; \text{ thus, } \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos 60^\circ = \frac{1}{2}$$

Substituting, we obtain:

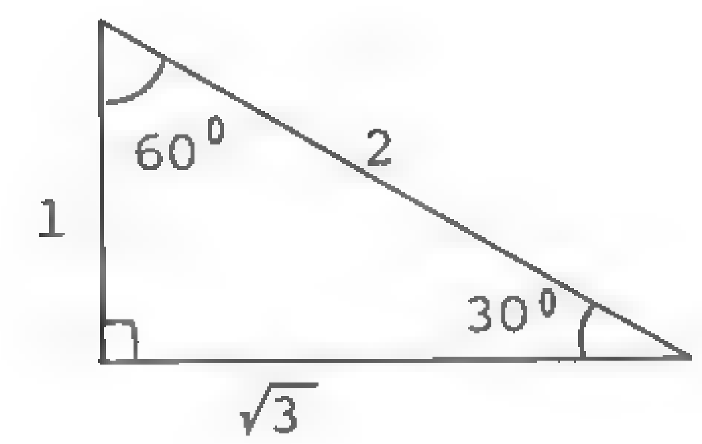
$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

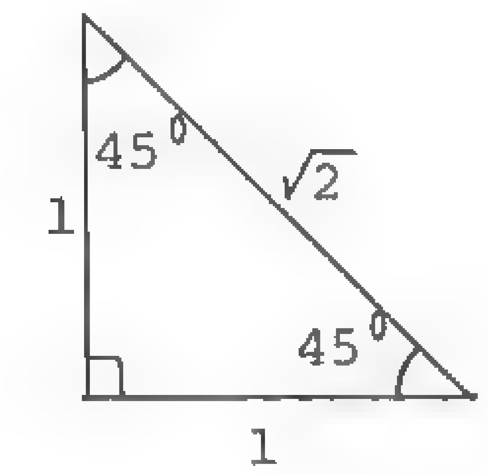
Multiply the fractions (recall $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$) to obtain $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6} + \sqrt{2}}{4}$. Therefore, $\sin 105^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$.

• PROBLEM 2-29

Find $\cos 15^\circ$, without the use of a Trig. table.





Solution: We express $\cos 15^\circ$ as the cosine of the difference of two angles whose cosine and sine we know. Since we know $15^\circ = 45^\circ - 30^\circ$, then $\cos 15^\circ = \cos(45^\circ - 30^\circ)$. Now we apply the formula for the cosine of the difference of two angles, which states:

$$\cos(u - v) = \cos u \cos v + \sin u \sin v .$$

Thus, $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$. Now, we must find the values for $\cos 45^\circ$, $\cos 30^\circ$, $\sin 30^\circ$. This can be accomplished by observing the 45-45 and 30-60 right triangles.

$$\text{Since } \cos = \frac{\text{adjacent}}{\text{hypothenuse}} \text{ we find: } \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2} ,$$
$$\text{and since } \sin = \frac{\text{opposite}}{\text{hypotenuse}} \text{ we find: } \sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin 30^\circ = \frac{1}{2}$$

Substituting, we obtain:

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} .$$

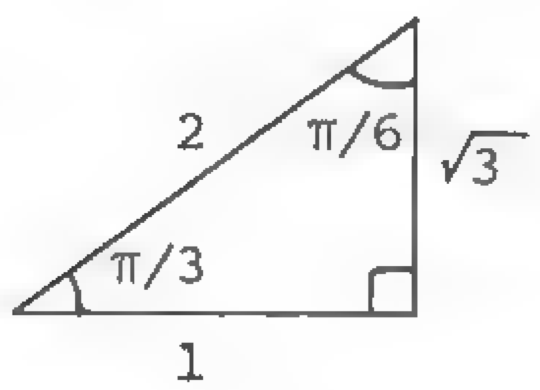
Multiplying the fractions (recall $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$) we obtain:

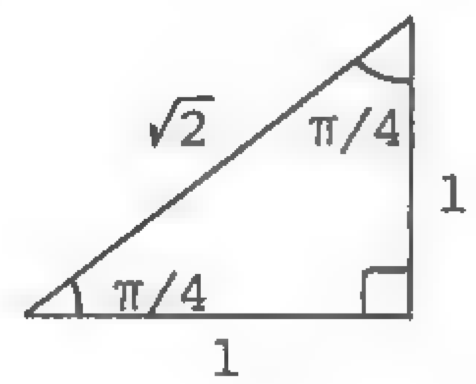
$$\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} .$$

Therefore, $\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$.

• PROBLEM 2-30

Find $\cos \frac{1}{12}\pi$.





<u>Solution:</u> Express $\cos \frac{\pi}{12}$ in terms of angles whose values of the trigonometric functions are known.

$$\cos \frac{1}{12}\pi = \cos\left(\frac{4}{12}\pi - \frac{3}{12}\pi\right)$$

$$= \cos\left(\frac{1}{3}\pi - \frac{1}{4}\pi\right)$$

Now apply the difference formula for the cosine of two angles, α and β. $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.

In this example, $\alpha = \frac{1}{3}\pi$ and $\beta = \frac{1}{4}\pi$.

$$\cos\left(\frac{1}{3}\pi - \frac{1}{4}\pi\right) = \cos\frac{1}{3}\pi \cos\frac{1}{4}\pi$$

$$+ \sin\frac{1}{3}\pi \sin\frac{1}{4}\pi$$

See the accompanying diagrams to find the values of these angles. We find:

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

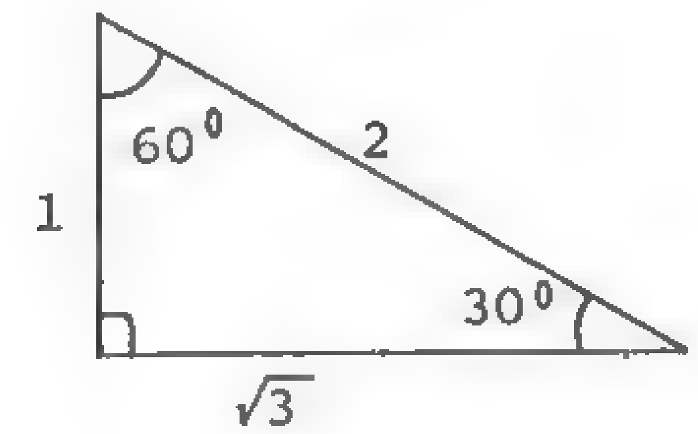
Thus, $\cos\left(\frac{1}{3}\pi - \frac{1}{4}\pi\right) = \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

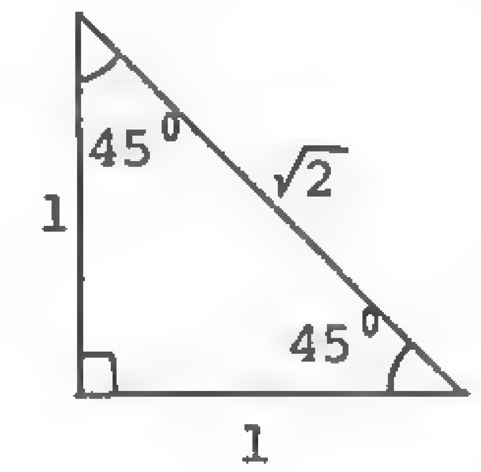
$$+ \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}\,\sqrt{3}}{4}$$

$$= \frac{1}{4}\sqrt{2}\,(1 + \sqrt{3})$$

• PROBLEM 2-31

Find the sine and cosine of 75°.





<u>Solution:</u> The angle 75° may be expressed as the sum or difference of two special angles, whose functions are known, in various ways: $75^\circ = 45^\circ + 30^\circ$, $75^\circ = 120^\circ - 45^\circ$, etc. If we choose the first of these, we use the sine of the sum of two angles,

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha .$$

Here $\alpha = 45^\circ$ and $\beta = 30^\circ$. Then,

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ .$$

To find the sine and cosine of 45° and 30°, construct 45°, 45°, 90° and 30°, 60°, 90° triangles (see Figures). Since $\sin(\measuredangle) = \frac{\text{side opposite}(\measuredangle)}{\text{hypotenuse}}$, and $\cos(\measuredangle) = \frac{\text{side adjacent }(\measuredangle)}{\text{hypotenuse}}$; then

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}, \quad \sin 30^\circ = \tfrac{1}{2},$$

and

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2} .$$

Substituting,

$$\sin 75^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}\sqrt{3}}{4} + \frac{\sqrt{2}}{4} = \frac{1}{4}\sqrt{6} + \frac{1}{4}\sqrt{2}$$

$$= \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right).$$

To find $\cos 75^\circ$, apply the formula for the cosine of the sum of two angles,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta .$$

Here $\alpha = 45^\circ$ and $\beta = 30^\circ$. Thus,

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ .$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \tfrac{1}{4}\left(\sqrt{6} - \sqrt{2}\right).$$

If we use the approximate values, $\sqrt{6} = 2.4494$, $\sqrt{2} = 1.4142$, we find

$$\sin 75^\circ = 0.9659, \quad \cos 75^\circ = 0.2588,$$

which check with the values given in the tables.

• PROBLEM 2-32

Find an expression for $\tan(u + v)$.

Solution: By definition of the tangent function, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Then,

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} \qquad (1)$$

The addition formulas for the sine and cosine functions are:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Replacing α by u and β by v, and using these addition formulas, equation (1) becomes:

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u\cos v + \cos u\sin v}{\cos u\cos v - \sin u\sin v}$$

If neither $\cos u = 0$ nor $\cos v = 0$, we can divide both the numerator and the denominator of this fraction by the product $\cos u\cos v$ to obtain a formula that involves only the tangent function:

$$\tan(u+v) = \frac{\dfrac{\sin u\cos v + \cos u\sin v}{\cos u\cos v}}{\dfrac{\cos u\cos v - \sin u\sin v}{\cos u\cos v}}$$

$$= \frac{\dfrac{\sin u\cancel{\cos v}}{\cos u\cancel{\cos v}} + \dfrac{\cancel{\cos u}\sin v}{\cancel{\cos u}\cos v}}{\dfrac{\cancel{\cos u}\,\cancel{\cos v}}{\cancel{\cos u}\,\cancel{\cos v}} - \dfrac{\sin u\sin v}{\cos u\cos v}}$$

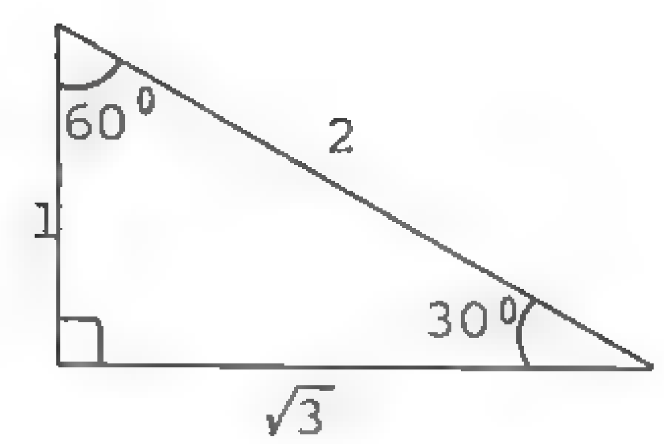
$$= \frac{\dfrac{\sin u}{\cos u} + \dfrac{\sin v}{\cos v}}{1 - \dfrac{\sin u\sin v}{\cos u\cos v}}$$

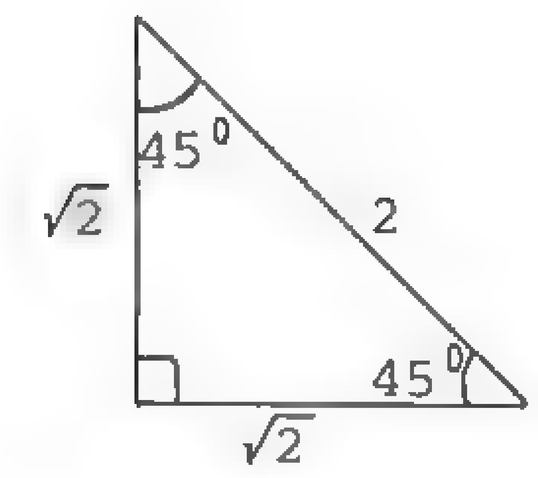
$$= \frac{\dfrac{\sin u}{\cos u} + \dfrac{\sin v}{\cos v}}{1 - \dfrac{\sin u}{\cos u}\dfrac{\sin v}{\cos v}}$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}.$$

• PROBLEM 2-33

Find $\sin 15^\circ$, $\cos 15^\circ$, $\tan 15^\circ$, and $\cot 15^\circ$.





Solution: To find the values of these trigonometric functions, use the following subtraction formulas for the sine, cosine, tangent, and cotangent functions.

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$$

Recall that in a $30^\circ - 60^\circ$ right triangle:

$$\sin 30^\circ = \tfrac{1}{2} \text{ and } \cos 30^\circ = \frac{\sqrt{3}}{2},$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2} \text{ and } \cos 60^\circ = \tfrac{1}{2}.$$

In a $45^\circ - 45^\circ$ right triangle:

$$\sin 45^\circ = \frac{\sqrt{2}}{2} \text{ and } \cos 45^\circ = \frac{\sqrt{2}}{2} \quad \text{(see figures)}$$

Now, substitute $\alpha = 45^\circ$ and $\beta = 30^\circ$ in the above formulas.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{2}\sqrt{3}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{2}\sqrt{3}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

$$\tan 15^\circ = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ\tan 30^\circ}$$

$$= \frac{1 - \dfrac{\sqrt{3}}{3}}{1 + 1\left(\dfrac{\sqrt{3}}{3}\right)} = \frac{1 - \dfrac{\sqrt{3}}{3}}{1 + \dfrac{\sqrt{3}}{3}}$$

Obtaining a common denominator of 3 for both the numerator and the denominator,

$$\tan 15^\circ = \frac{\frac{3}{3} - \frac{\sqrt{3}}{3}}{\frac{3}{3} + \frac{\sqrt{3}}{3}} = \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}} = \frac{3-\sqrt{3}}{\cancel{3}} \cdot \frac{\cancel{3}}{3+\sqrt{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}}$$

$$\cot 15^\circ = \cot(45^\circ - 30^\circ)$$

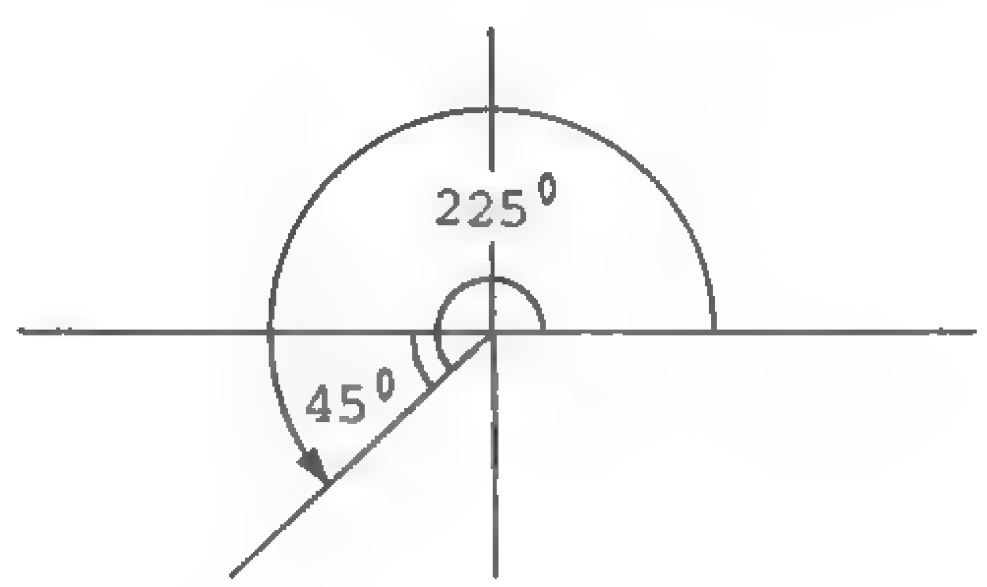
$$= \frac{\cot 45^\circ \cot 30^\circ + 1}{\cot 30^\circ - \cot 45^\circ}$$

$$= \frac{(1)(\sqrt{3}) + 1}{\sqrt{3} - 1}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

• PROBLEM 2-34

Find the trigonometric functions of 195° using functions of 225° and 30°.



Solution: $195^\circ = 225^\circ - 30^\circ$. Hence, $\sin 195^\circ = \sin(225^\circ - 30^\circ)$. Using the subtraction formula for sines, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ where $\alpha = 225^\circ$ and $\beta = 30^\circ$, $\sin 195^\circ = \sin(225^\circ - 30^\circ) = \sin 225^\circ \cos 30^\circ - \cos 225^\circ \sin 30^\circ$.
The reference angle for 225° is $45^\circ(225^\circ - 180^\circ = 45^\circ)$, and the angle 225° is a third quadrant angle. See the figure.

In the third quadrant both the sine and cosine functions are negative. From a table of trigonometric functions, $\sin 225^\circ = \sin 45^\circ = -0.7071 = -\frac{\sqrt{2}}{2}$, $\cos 225^\circ = \cos 45^\circ = -0.7071 = -\frac{\sqrt{2}}{2}$, $\cos 30^\circ = 0.8660 = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = 0.5000 = \frac{1}{2}$. Therefore,

$$\sin 195^\circ = \sin(225^\circ - 30^\circ) = \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{-\sqrt{2}\sqrt{3}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{-\sqrt{2}\sqrt{3} + \sqrt{2}}{4}$$

$$= \frac{\sqrt{2}(-\sqrt{3} + 1)}{4}$$

Hence, $\sin 195^\circ = \frac{\sqrt{2}(1 - \sqrt{3})}{4}$.

• PROBLEM 2-35

Derive a formula for $\sin 3\alpha$ in terms of $\sin\alpha$.

Solution: We may regard 3α as $\alpha + 2\alpha$, and use the addition formula for the sine of two angles.

$$\sin(a + b) = \sin a \cos a + \cos a \sin b$$

where $a = \alpha$ and $b = 2\alpha$.

$$\sin 3\alpha = \sin(\alpha + 2\alpha) = \sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha .$$

Now replace $\sin 2\alpha$ and $\cos 2\alpha$ by the expressions

$$\cos 2a = \cos^2 a - \sin^2 a$$

and

$$\sin 2a = 2 \sin a \cos a .$$

We find that

$$\sin 3\alpha = (\sin\alpha)(\cos^2\alpha - \sin^2\alpha) + (\cos\alpha)(2\sin\alpha\cos\alpha)$$

$$= \sin\alpha\cos^2\alpha - \sin^3\alpha + 2\sin\alpha\cos^2\alpha$$

$$= 3\sin\alpha\cos^2\alpha - \sin^3\alpha .$$

Finally, since we wish a result involving only $\sin\alpha$, replace $\cos^2\alpha$ by $1 - \sin^2\alpha$; then

$$\sin 3\alpha = 3(\sin\alpha)(1 - \sin^2\alpha) - \sin^3\alpha$$

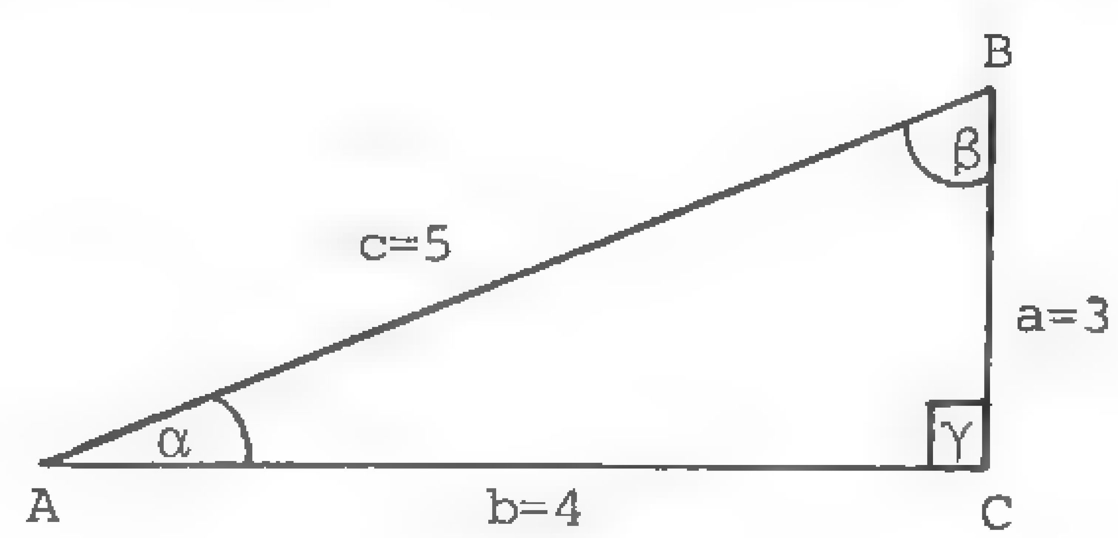
$$= 3\sin\alpha - 3\sin^3\alpha - \sin^3\alpha$$

$$= 3\sin\alpha - 4\sin^3\alpha .$$

This is the desired identity.

• PROBLEM 2-36

Given the right triangle with $a = 3$, $b = 4$, and $c = 5$, find the values of the trigonometric functions of α.



Solution: In the accompanying figure, a is the side opposite angle α, b is the side opposite angle β, and c is the side opposite angle γ. The values of the trigonometric functions of α are:

$$\cos\alpha = \frac{\text{adjacent side}}{\text{hypotenuse}}, \quad \sin\alpha = \frac{\text{opposite side}}{\text{hypotenuse}},$$

$$\tan\alpha = \frac{\text{opposite side}}{\text{adjacent side}}, \quad \cot\alpha = \frac{1}{\tan\alpha},$$

$$\sec\alpha = \frac{1}{\cos\alpha}, \quad \text{and} \quad \csc\alpha = \frac{1}{\sin\alpha}.$$

Therefore:

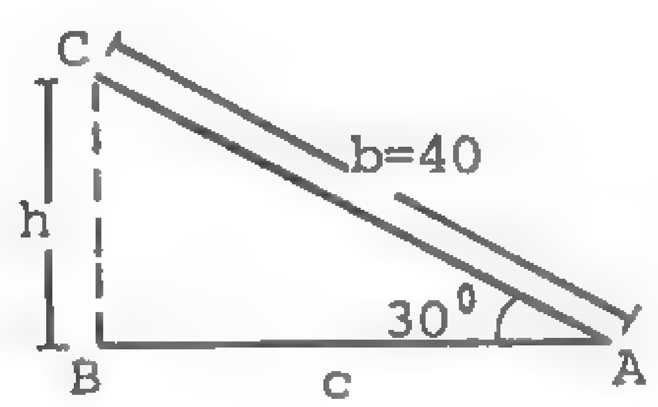
$$\cos\alpha = \frac{4}{5}, \quad \sin\alpha = \frac{3}{5},$$

$$\tan\alpha = \frac{3}{4}, \quad \cot\alpha = \frac{1}{3/4} = \frac{4}{3},$$

$$\sec\alpha = \frac{1}{4/5} = \frac{5}{4}, \quad \csc\alpha = \frac{1}{3/5} = \frac{5}{3}.$$

• PROBLEM 2-37

Solve the oblique triangle ABC for side c, and the two unknown angles, where $a = 20$, $b = 40$, $\alpha = 30$; and α is the angle between sides b and c.



Solution: If we draw an altitude h from b to c, as in the accompanying diagram, we can find the length of this altitude by trigonometry.

$$\sin 30° = \frac{h}{b}$$

$$h = b\sin 30°$$

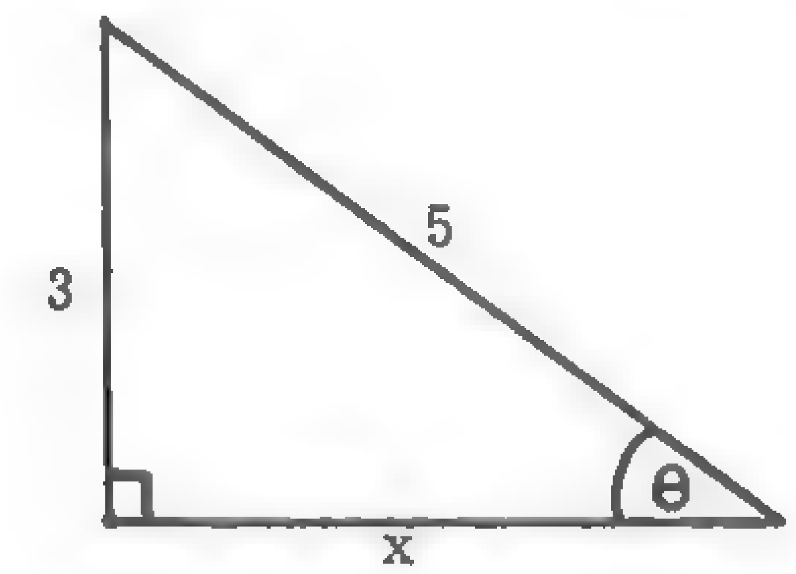
Since $\sin 30° = \frac{1}{2}$, and $b = 40$,

$$h = 40\left(\tfrac{1}{2}\right) = 20 = \text{side } a$$

Thus, the triangle must have the altitude as one of its sides; therefore, we have a right triangle with angles 30°, 60°, and 90°. Sides of such right triangles are in proportion $1 : \sqrt{3} : 2$, and the lengths are therefore $20 : 20\sqrt{3} : 40$. Hence, $c = 20\sqrt{3}$, and the two unknown angles are 60° and 90°.

• PROBLEM 2-38

Find the values of the six trigonometric functions of an angle, in a right triangle, whose opposite side is 3 and hypotenuse 5.



Solution: We are given that, in a right triangle, the side opposite the angle is 3 and the hypotenuse is 5. To determine the adjacent side, x (see figure), we recall the Pythagorean Theorem which states that the sum of the square of the legs of a right triangle equals the square of the hypotenuse. Thus,

$$3^2 + x^2 = 5^2$$

$$9 + x^2 = 25$$

$$x^2 = 16$$

$$x = 4$$

Now that we know the value of each side of the triangle, we can find the values of the six trigonometric functions:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{3}{4}$$

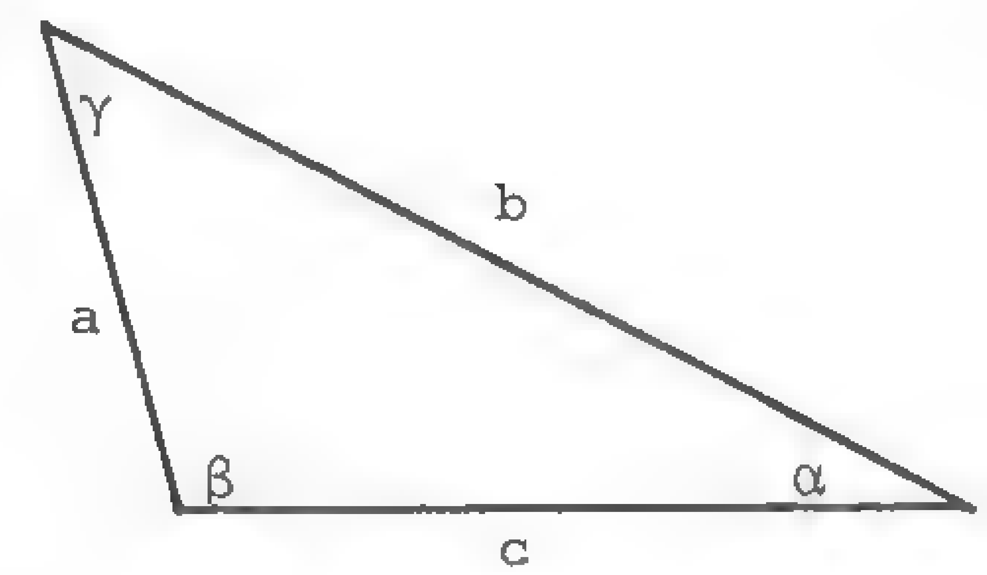
$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{5}{3}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{5}{4}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{4}{3}$$

• PROBLEM 2-39

Given $a = 8$, $c = 7$, $\beta = 135°$, find b.



Solution: Use the law of cosines to find one side given two sides and an included angle.

$b^2 = a^2 + c^2 - 2ac \cos(\text{included angle})$ where the included angle is the angle between the two given sides.

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$b^2 = 64 + 49 - 2 \cdot 8 \cdot 7 \cos 135°$$

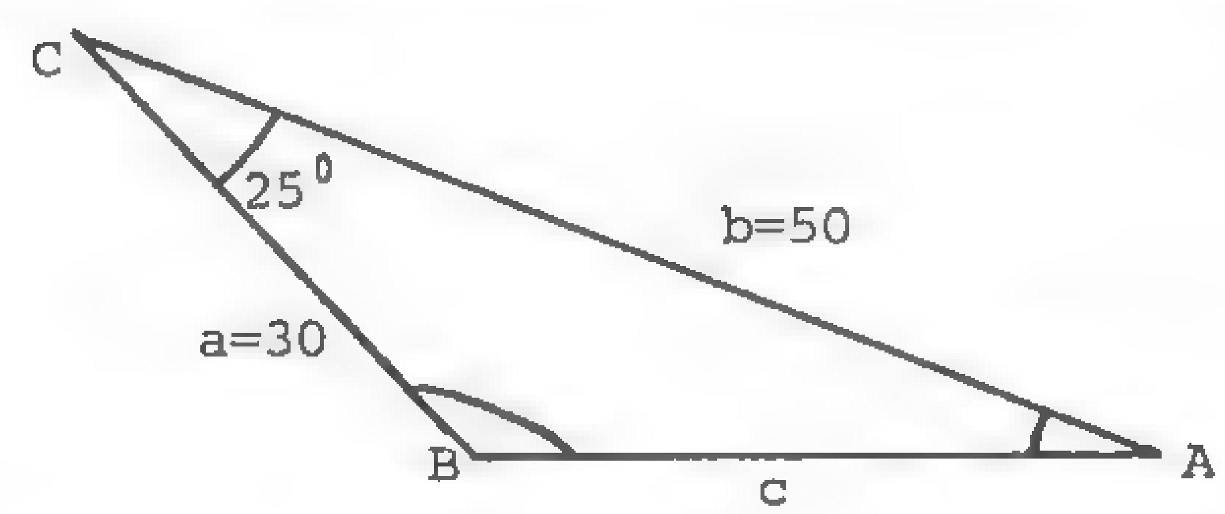
$$= 113 - 112 \cdot \left(\frac{\sqrt{2}}{2}\right) = 113 + 79.184$$

$$= 192.184$$

$$b = 13.863$$

• PROBLEM 2-40

Solve triangle ABC, given $a = 30$, $b = 50$, $\angle C = 25°$.



Solution: Two of the sides of ΔABC, and their included angle are given. We wish to find the third side, c. Therefore use the law of cosines to find c.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 30^2 + 50^2 - 2(30)(50) \cos 25°$$

$$c^2 = 900 + 2500 - 2(30)(50)(0.9063)$$

$$c^2 = 681.1$$

$$c = 26 \text{ (to two significant digits)}$$

Use the law of sines to find one of the remaining angles.

$$\frac{\sin A}{30} = \frac{\sin 25°}{26}$$

$$\sin A = \frac{30 \sin 25°}{26} = \frac{30(0.4226)}{26}$$

$$\sin A = 0.4876$$

$$\angle A = 29° \text{ (to the nearest degree)}$$

$\angle B$ can be found from $\angle A$ and $\angle C$.

$$\angle A + \angle B + \angle C = 180°$$

$$B = 180° - \angle A - \angle C$$

$$= 180° - (\angle A + \angle C)$$

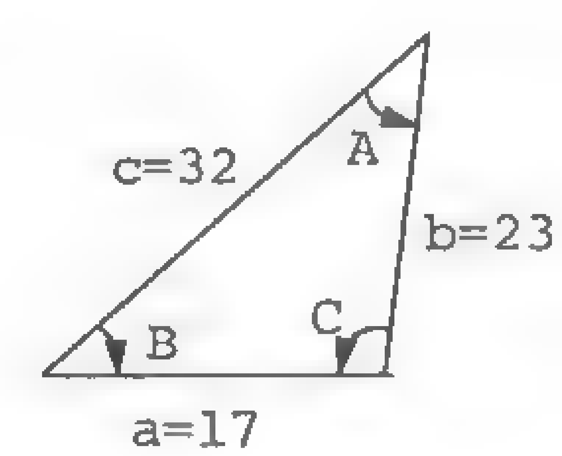
$$= 180° - (29° + 25°)$$

$$= 180° - (54°)$$

$$\angle B = 126°$$

• PROBLEM 2-41

Solve the triangle ABC, given $a = 17$, $b = 23$, $c = 32$.



Solution: Use the generalized Pythagorean theorem to determine the angles A, B, C. The angles appear in the upper case directly opposite the corresponding lower case designation of the side, (see figure).

The Law of cosines (generalized Pythagorean) states $z^2 = x^2 + y^2 - 2xy \cos(\text{included angle})$. The included angle is the angle between the two known sides, x and y. This problem is a variation of the usual Law of Cosines problem,which gives values of 2 sides and their included angle, and instead supplies the three sides and asks for the angles opposite those sides Solve the equation for cos (included angle) to obtain:

$$\cos(\text{included angle}) = \frac{x^2 + y^2 - z^2}{2xy}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos A = \frac{23^2 + 32^2 - 17^2}{(2)(23)(32)} \qquad \cos B = \frac{17^2 + 32^2 - 23^2}{(2)(17)(32)}$$

$$\cos A = 0.8587 \qquad \cos B = 0.7206.$$

$\angle A = 31°$ (nearest degree) $\quad \angle B = 44°$ (nearest degree)

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{17^2 + 23^2 - 32^2}{(2)(17)(23)}$$

$$\cos C = -\,0.2634$$

$\angle C = 105°$ (nearest degree)

It is not necessary to calculate angle C using the law of cosines when angles A and B are known. Angle C can be found instead by the following:

$\angle A + \angle B + \angle C = 180°$ since the sum of the angles of a triangle is 180°,

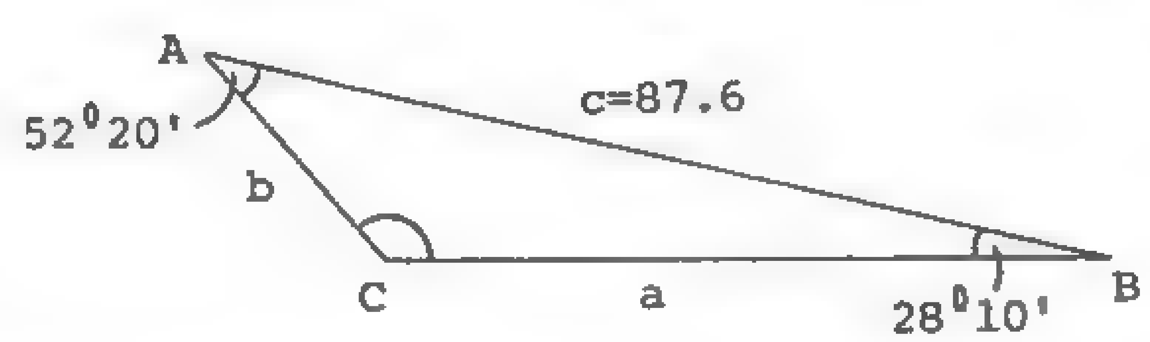
$$\text{or } 31° + 44° + \angle C = 180°$$

$$75° + \angle C = 180°$$

$$\angle C = 180° - 75° = 105°.$$

• PROBLEM 2-42

> Solve triangle ABC for $\angle C$, a, and b, given
> $\angle A = 52°20'$, $\angle B = 28°10'$, $c = 87.6$



Solution: Given two angles, the third is determined, since the three angles of a triangle equal 180°. Thus,

$$\angle A + \angle B + \angle C = 180°.$$

$$\angle C = 180° - (\angle A + \angle B)$$

$$\angle C = 180° - 80°30' = 179°60' - 80°30' = 99°30'.$$

Sides a and b can be determined from the law of sines. In any tri-

angle, the sides are proportional to the sines of the opposite angles (see figure). Then:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

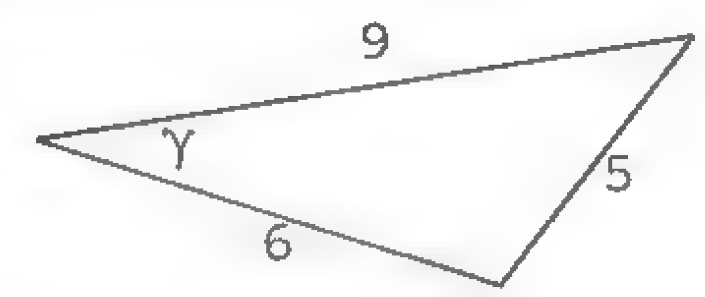
Now, since $\angle A$, $\angle B$, and side C were given, and we previously determined $\angle C$, substituting into these proportions, and using a table of trigonometric functions, will give us a and b. Thus,

$$a = \frac{c \sin A}{\sin C} = \frac{c \sin 52°20'}{\sin 99°30'} \qquad b = \frac{c \sin B}{\sin C} = \frac{c \sin 28°10'}{\sin 99°30'}$$

$$a = \frac{87.6(0.7916)}{(0.9863)} = 70.3 \qquad b = \frac{87.6(0.4720)}{(0.9863)} = 41.9$$

• PROBLEM 2-43

> Given $a = 9$, $b = 6$, $c = 5$, find γ.



Solution: Since the lengths of the three sides of the triangle are known, the only unknown is $\cos\gamma$. If this value is known, the value of γ can be found. Since triangles can have angles no larger than 180°, if $\cos\gamma$ is positive, γ is a first quadrant angle, and if $\cos\gamma$ is negative, γ is a second quadrant angle. Use the law of cosines to determine the angle since three sides are given, i.e., $c^2 = a^2 + b^2 - 2ab\cos(\text{included angle})$. Here all is known except the included angle.

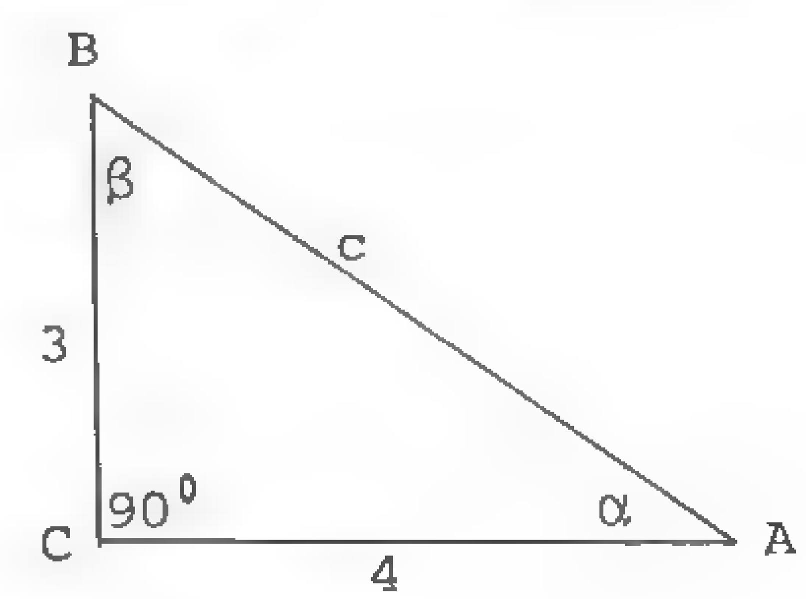
$$5^2 = 9^2 + 6^2 - 2 \cdot 9 \cdot 6 \cos\gamma$$

$$\cos\gamma = \frac{25 - 81 - 36}{-108} = \frac{92}{108} = 0.85185$$

$$\gamma = 31°35.2'$$

• PROBLEM 2-44

> In right triangle ABC, if $a = 3$, and $b = 4$, find c, α, β.



Solution: The figure can be drawn with accuracy in this case. (See the Figure).

The Pythagorean Theorem $c^2 = a^2 + b^2$ such that $c^2 = 3^2 + 4^2 = 25$ shows that $c = 5$. The ratio of the two known sides is, indeed, a value of a trigonometric function of one of the acute angles. In fact,

$$\tan\alpha = \frac{\text{side opposite} \measuredangle \alpha}{\text{side adjacent} \measuredangle \alpha} = \frac{3}{4}$$

Similarly for $\measuredangle\beta$, $\tan\beta = \frac{4}{3}$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\dfrac{\text{side opposite} \measuredangle\alpha}{\text{side adjacent} \measuredangle\alpha}} = \frac{\text{side adjacent} \measuredangle\alpha}{\text{side opposite} \measuredangle\alpha} = \frac{4}{3}$$

Similarly for $\measuredangle\beta$, $\cot\beta = \frac{3}{4}$. It is merely a matter of choosing the preferred one of these ratios. Let us choose $\tan\alpha = \frac{3}{4}$. Thus,

$$\tan\alpha = \frac{3}{4} = 0.75000$$

$$\alpha = 36^\circ 52.2'$$

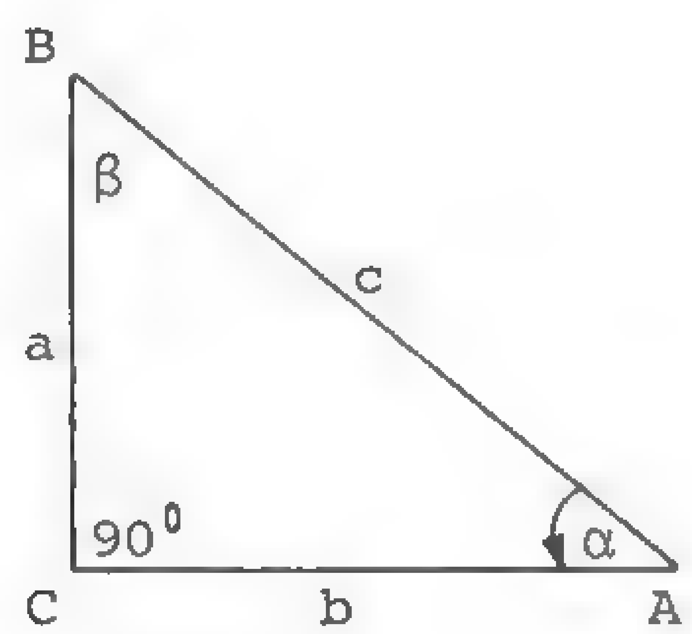
$$\alpha + \beta + 90^\circ = 180^\circ$$

$$\alpha + \beta = 180^\circ - 90^\circ = 90^\circ$$

Therefore, $$\beta = 90^\circ - \alpha = 90^\circ - 36^\circ 52.2' = 53^\circ 7.8'$$

• PROBLEM 2-45

In triangle ABC, if $a = 675$, $\alpha = 48^\circ 36'$, find b, c, β.



Solution: If we wish to work with α, the functions of α involving a and one other side are

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

$\sin\alpha = a/c$

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$

$\tan\alpha = a/b$, and

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{\text{adjacent}}{\text{opposite}}$$

$\cot\alpha = b/a$. Either of these ratios can be chosen. If the first is chosen, $\sin\alpha = a/c$ or $c \sin\alpha = a$ and $c = a/\sin\alpha$. Thus,

$$c = \frac{675}{\sin 48^\circ 36'} = \frac{675}{0.75011} = 899.85$$

Using $\tan\alpha = a/b$ or $b = a/\tan\alpha$, we obtain

$$b = \frac{675}{\tan 48^\circ 36'} = 595.08$$

To find β, $\alpha + \beta + 90^\circ = 180^\circ$

$$\alpha + \beta = 180^\circ - 90^\circ = 90^\circ$$

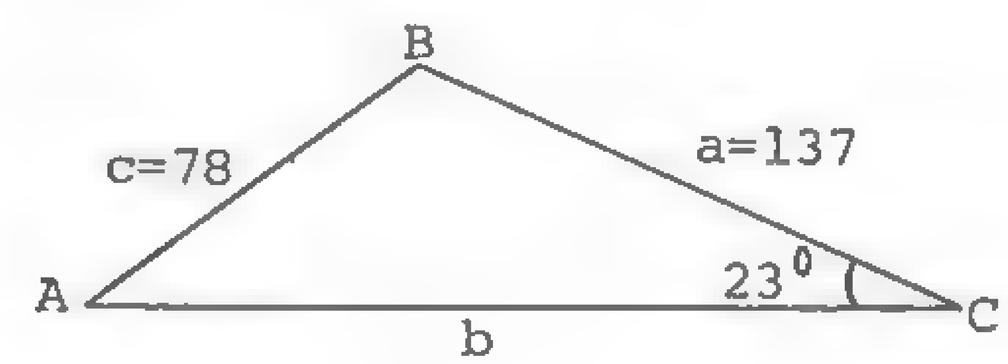
$$\beta = 90^\circ - \alpha = 90^\circ - 48^\circ 36'$$

$$\beta = 41^\circ 24'$$

• PROBLEM 2-46

Find all the sides and angles of triangle ABC, given $a = 137$, $c = 78.0$, $< C = 23^\circ 0'$.

Solution: Draw triangle ABC, filling in the given information. Thus



we divide the problem into 3 parts:

(1) find angle A.

(2) find angle B.

(3) find side b.

In order to find angle A we may use the law of sines, $\frac{\sin A}{\text{side } a} = \frac{\sin C}{\text{side } c}$,

because we are given side $a = 137$, side $c = 78$, and $\sin C = \sin 23^\circ$; thus, $\frac{\sin A}{137} = \frac{\sin 23^\circ 0'}{78}$.

Using our Trig. table we find $\sin 23^\circ = 0.3907$. Thus

$$\frac{\sin A}{137} = \frac{0.3907}{78}$$

Multiplying both sides by 137 we obtain

$$\sin A = \frac{137(0.3907)}{78}$$

$$\sin A = 0.6862 .$$

Using our Trig. table we find that $\sphericalangle A = 43^\circ 20'$. Our Trig. table only gives values of sine between 0° and 90° . Since we are dealing with an angle in a triangle, which can take on values greater than 90° (recall there are 180° in a triangle) we must examine what happens to the sine function in the second quadrant, that is between 90° and 180°

We use our trignometric identity $\sin\theta = \sin(180-\theta)$:

$$\sin 43^\circ 20' = \sin(180-43^\circ 20') = \sin 136^\circ 40'$$

Hence $\quad \sphericalangle A = 43^\circ 20'$ <u>or</u> $136^\circ 40'$

Thus there are two solutions. We now proceed to the next part of our problem, finding angle B, and side b.

$$\sphericalangle A + \sphericalangle B + \sphericalangle C = 180^\circ \quad \text{(There are } 180^\circ \text{ in a triangle)}$$

$$\sphericalangle A + \sphericalangle B + 23^\circ = 180^\circ$$

$$\sphericalangle A + \sphericalangle B = 157^\circ$$

If $\sphericalangle A = 43^\circ 20'$ then

$$43^\circ 20' + \sphericalangle B = 157^\circ$$

$$\sphericalangle B = 157^\circ - 43^\circ 20' = 113^\circ 40'$$

Since we now know $\sphericalangle B$, we may apply the law of sines:

$$\frac{\sin B}{\text{side } b} = \frac{\sin C}{\text{side } c}$$

to find side b:

$$\frac{\sin 113^\circ 40'}{b} = \frac{\sin 23^\circ}{78}$$

Cross multiplying we obtain

$$b \sin 23^\circ = 78 \sin 113^\circ 40'$$

$$b = \frac{78 \sin 113^\circ 40'}{\sin 23^\circ}$$

Substituting in the values $\sin 113^\circ 40' = 0.9159$ and $\sin 23^\circ = 0.3907$ we obtain

$$b = 78\left(\frac{0.9159}{0.3907}\right) = 183$$

Hence if we choose $< A = 43^\circ 20'$ then

$$< B = 113^\circ 40' \text{ and}$$

$$\text{side } b = 183$$

If, however, we choose $< A = 136^\circ 40'$, then since

$$< A + < B = 157^\circ$$

$$136^\circ 40' + < B = 157^\circ$$

and $$< B = 157^\circ - 136^\circ 40' = 20^\circ 20'$$

Applying the law of sines to find side b gives us:

$$\frac{\sin B}{\text{side } b} = \frac{\sin C}{\text{side } c}$$

$$\frac{\sin 20^\circ 20'}{b} = \frac{\sin 23^\circ}{78}$$

Cross multiplying gives us

$$b \sin 23^\circ = 78 \sin 20^\circ 20'$$

$$b = \frac{78 \sin 20^\circ 20'}{\sin 23^\circ}$$

Substituting in $\sin 20^\circ 20' = 0.3475$ and $\sin 23^\circ = 0.3907$ we obtain

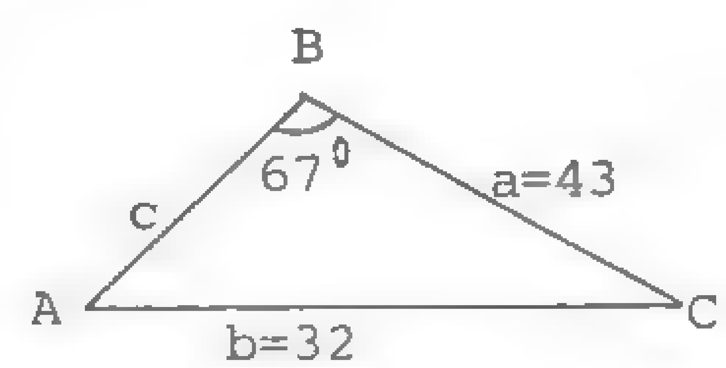
$$b = \frac{78(0.3475)}{0.3907} = 69$$

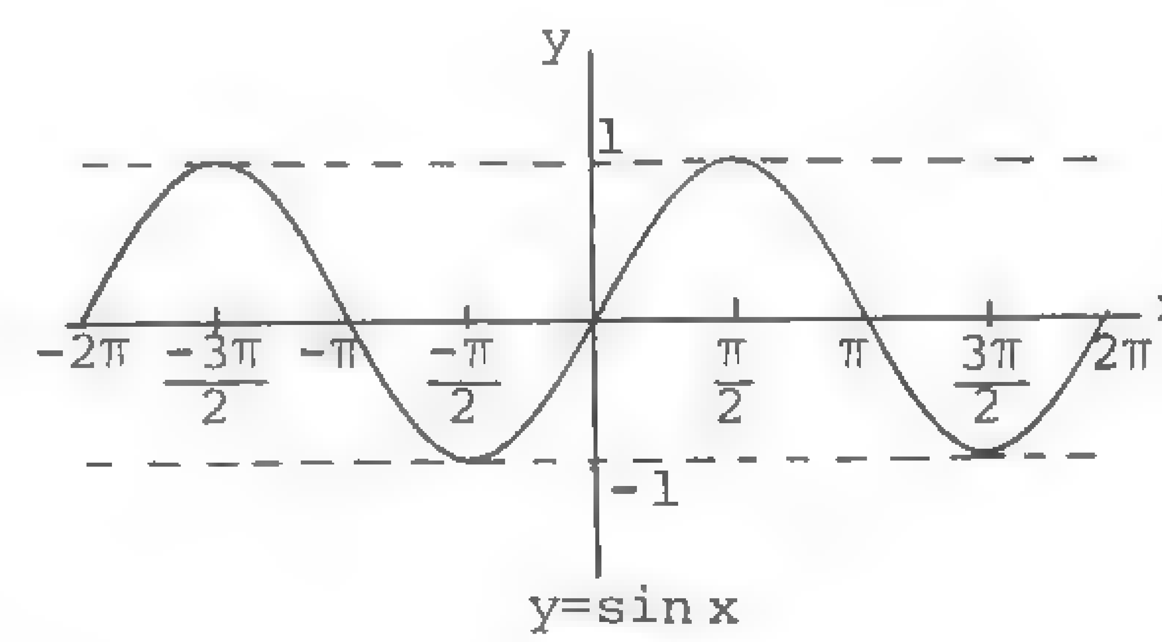
Hence if we choose $< A = 136^\circ 40'$, then $< B = 20^\circ 20'$, and side $b = 69$.

• PROBLEM 2-47

Find all the sides and angles of triangle ABC, given $a = 43$, $b = 32$, $< B = 67^\circ$.

<u>Solution:</u> Draw triangle ABC, filling in the given information. Thus





we divide the problem into 3 parts:

(1) find angle A

(2) find angle C

(3) find side c

In order to find angle A we may use the law of sines, $\frac{\sin A}{\text{side } a} = \frac{\sin B}{\text{side } b}$, because we are given, side $a = 43$, side $b = 32$, and $\sin B = \sin 67^\circ$. Thus, $\frac{\sin A}{43} = \frac{\sin 67}{32}$, multiplying both sides by 43,

$$\sin A = \frac{43 \sin 67^\circ}{32}$$

In our trig table we find $\sin 67^\circ = 0.9205$. Thus,

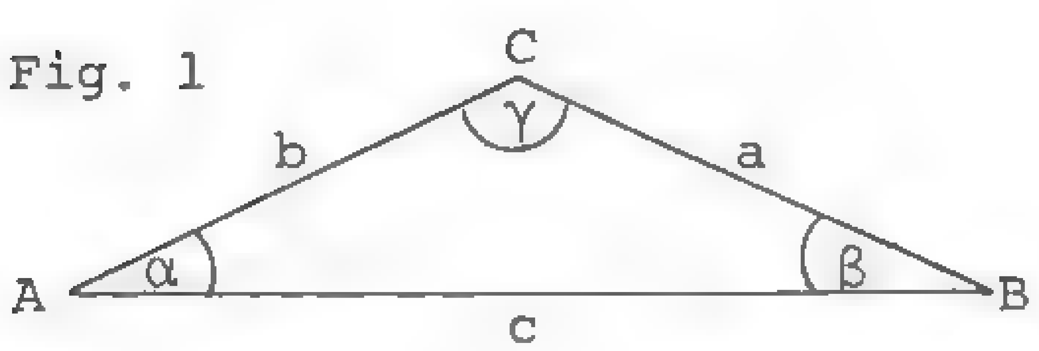
$$\sin A = \frac{43(0.9205)}{32} = 1.2369$$

We found $\sin A = 1.2369$; however the sine function is only defined on the interval $[-1,1]$ — that is, it can only take on values between -1 and 1, which can be seen from the graph $y = \sin x$, as shown. Thus there is no angle A such that $\sin A = 1.2369$, and no triangle can contain such an angle, hence triangle ABC is non-existent, and we cannot solve for the other sides or angles.

• PROBLEM 2-48

Solve the given oblique triangle ABC for a, c and β, where $b = 47$, $\alpha = 20°43'$, $\gamma = 153°44'$.

Fig. 1



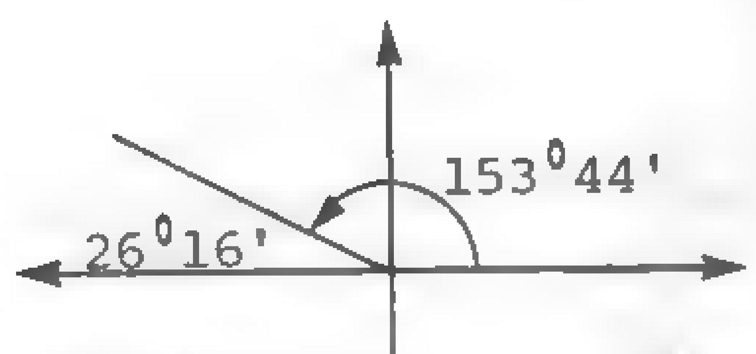


Fig. 2

Solution: The given triangle can be seen in Figure 1. Since the sum of the angles in a triangle is 180°, $\alpha + \beta + \gamma = 180°$, or $\beta = 180° - \alpha - \gamma$; and substituting:

$$\begin{aligned}\beta &= 180° - 20°43' - 153°44' \\ &= (179°60' - 20°43') - 153°44' \\ &= 159°17' - 153°44' \\ &= 158°77' - 153°44' = 5°33'\end{aligned}$$

Using the law of sines, we set up the proportion,

$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$. Since we are given α and b, and have just found β, we employ this law to obtain a. We have:

$$\frac{a}{\sin 20°43'} = \frac{47}{\sin 5°33'}$$

$$a = \sin 20°43'\left(\frac{47}{\sin 5°33'}\right)$$

Taking the logarithm of both sides of this equation, and recalling the fact that $\log(a/b) = \log a - \log b$, and $\log(ab) = \log a + \log b$, we obtain:

$$\log a = \log \sin 20°43' + (\log 47 - \log \sin 5°33').$$

To solve for log a we use a table of logs of trig functions, a table of logs of numbers, and interpolation. First let us find log sin 20°43'. To do this we interpolate:

$$\begin{array}{cc|cc|c} & & x & \log \sin x & \\ \hline 10 & 3 & 20°40' & 9.5477 - 10 & y \quad .0033 \\ & & 20°43' & ? & \\ & & 20°50' & 9.5510 - 10 & \end{array}$$

$$\frac{3}{10} = \frac{y}{.0033}$$

$$10y = .0099$$

$$y = .00099$$

$$\log \sin 20°43' = 9.5477 - 10 + .00099 = 9.54869 - 10$$

Now, we must find log 47. From a table of logarithms of numbers we find the mantissa to be 67210. To find the characteristic, recall that for a number greater than 1, the characteristic is positive and is one less than the number of digits before the decimal. Since the number 47 has two digits before the decimal, the characteristic is 1. Thus, $\log 47 = 1.67210$.

To obtain log sin 5°33' we again interpolate:

$$\begin{array}{cc|cc|c} & & x & \log \sin x & \\ \hline 10 & 3 & 5°30' & 8.9816 - 10 & y \quad .0129 \\ & & 5°33' & ? & \\ & & 5°40' & 8.9945 - 10 & \end{array}$$

$$\frac{3}{10} = \frac{y}{.0129}$$

$$10y = .0387$$

$$y = .00387$$

$$\log \sin 5°33' = 8.9816 - 10 + .00387 = 8.98547 - 10$$

Thus, substituting we obtain:

$$\log a = 9.54869 - 10 + [1.67210 - (8.98547 - 10)],$$

and since 1.67210 can be rewritten as 11.67210 - 10,

$$\log a = 9.54869 - 10 + [11.67210 - 10 - (8.98547 - 10)];$$

performing the subtraction in the brackets:

$$\begin{array}{r} 11.67210 - 10 \\ -\ (8.98547 - 10) \\ \hline 2.68663 \end{array} \quad = 12.68663 - 10$$

performing the addition:

$$\begin{array}{r} 9.54869 - 10 \\ +\ (12.68663 - 10) \\ \hline 22.23532 - 20 = 2.23532 \end{array}$$

Thus, log a = 2.23532

To find a we interpolate; observe that the characteristic is 2, and the mantissa is 23532. Recall that if the characteristic is positive, it is one less than the number of digits before the decimal in the number. Thus, there are 3 digits before the decimal, and interpolating:

$$\begin{array}{c} \\ 1 \left[\begin{array}{l} y \left[\begin{array}{ll} 171. & 2.2330 \\ a & 2.23532 \end{array} \right] .00232 \\ \quad\ 172. \qquad 2.2355 \end{array} \right] .0025 \end{array}$$

(columns: $\underline{N}$, $\underline{\log N}$)

$$\frac{y}{1} = \frac{.00232}{.0025}$$

$$y = \frac{232}{250} = .928$$

$$a = 171. + .928 = 171.928 \simeq 171.93$$

To find side c we again employ the law of sines, and set up the following proportion:

$\dfrac{c}{\sin \gamma} = \dfrac{b}{\sin \beta}$, and substituting:

$$\frac{c}{\sin 153°44'} = \frac{47}{\sin 5°33'}$$

$c = \sin 153°44' \left(\dfrac{47}{\sin 5°33'}\right)$. Taking the logarithm of both sides, we obtain:

$$\log c = \log \sin 153°44' + (\log 47 - \log \sin 5°33')$$

To solve for log c we must find log sin 153°44', and substitute this value, and those found previously for log 47 and log sin 5°33'. To obtain sin 153°44'

we first observe the coordinate axis, (see fig. 2), and discover that sin 153°44' = sin 26°16'. We now use interpolation:

$$10 \left[\begin{array}{l} 6 \left[\begin{array}{ll} 26°10' & 9.6444 - 10 \\ 26°16' & ? \end{array} \right] y \\ \quad\ 26°20' \qquad 9.6470 - 10 \end{array} \right] .0026$$

(columns: $\underline{x}$, $\underline{\log \sin x}$)

$$\frac{6}{10} = \frac{y}{.0026}$$

$$10y = .0156$$

$$y = .00156$$

$$\log \sin 26°16' = \log \sin 153°44' = 9.6444 - 10 + .00156$$
$$= 9.64596 - 10$$

Thus, since log 47 was found to equal 1.67210 and log sin 5°33' = 8.98547 - 10, we have:

$$\begin{aligned} \log c &= 9.64596 - 10 + [1.67210 - (8.98547 - 10)] \\ &= 9.64596 - 10 + [11.67210 - 10 - (8.98547 - 10)] \\ &= 9.64596 - 10 + (2.68663) \\ &= 9.64596 - 10 + (12.68663 - 10) \\ &= 22.33259 - 20 \end{aligned}$$

$$\log c = 2.33259$$

Interpolating, to find c:

$$1 \left[\begin{array}{l} y \left[\begin{array}{ll} 215. & 2.3324 \\ ? & 2.33259 \end{array} \right] .00019 \\ \quad\ 216. \qquad 2.3345 \end{array} \right] .0021$$

(columns: $\underline{x}$, $\underline{\log x}$)

$$y = \frac{.00019}{.0021} = \frac{19}{210} = .0904761$$

Thus, c = 215 + .090 = 215.09

Therefore,
a = 171.93
c = 215.09
β = 5°33'

CHAPTER 3

INVERSE TRIGONOMETRIC FUNCTIONS

• PROBLEM 3-1

Calculate the following numbers.

(a) Arctan $\sqrt{3}$ (c) Tan^{-1} 1.871

(b) Tan^{-1}.2027

Solution: a) The expression $\tan y = x$ is equivalent to $\arctan x = \tan^{-1} x = y$. Let the expression $\arctan \sqrt{3} = y$. Hence, the expression $\arctan \sqrt{3} = y$ is equivalent to $\tan y = \sqrt{3} = 1.7321$. In a table of trigonometric functions, the number y that corresponds to $\tan y = 1.7321$ is approximately 1.05.

b) Note that the expression $\tan^{-1}.2027 = \arctan .2027$. Let the expression $\tan^{-1} .2027 = y$. Hence, the expression $\tan^{-1}.2027 = \arctan .2027 = y$ is equivalent to $\tan y = .2027$. In a table of trigonometric functions, the number y that corresponds to $\tan y = .2027$ is .20.

c) Note that the expression $\tan^{-1} 1.871 = \arctan 1.871$. Let the expression $\tan^{-1} 1.871 = y$. Hence, the expression $\tan^{-1} 1.871 = \arctan 1.871 = y$ is equivalent to $\tan y = 1.871$. In a table of trigonometric functions, the number y that corresponds to $\tan y = 1.871$ is 1.08.

• PROBLEM 3-2

Find arcsin $\frac{1}{2}$.

Solution: Let $y = \arcsin \frac{1}{2}$; then $\sin y = \frac{1}{2}$. Since $\sin 30^\circ = \frac{1}{2}$, we have $y = 30^\circ, 150^\circ$. Because the sine has a period of 360°, any angle which we can obtain by adding any integral multiple of 360° to 30° or 150° would also satisfy this equation. We thus get

$$\arcsin \tfrac{1}{2} = \begin{cases} 2n\pi + \frac{1}{6}\pi \\ 2n\pi + \frac{5}{6}\pi \end{cases} \text{or} \begin{cases} n(360^\circ) + 30^\circ \\ n(360^\circ) + 150^\circ \end{cases},$$

where n may be any integer, positive, negative, or zero.

• PROBLEM 3-3

Find arctan $(-\sqrt{3})$.

Solution: $y = \arctan(-\sqrt{3})$; $\tan y = -\sqrt{3}$; $y = 120^\circ, 300^\circ$. Since the tangent has period π, we have

$$\arctan(-\sqrt{3}) = n\pi + \frac{2}{3}\pi .$$

• PROBLEM 3-4

Does the sine function have an inverse?

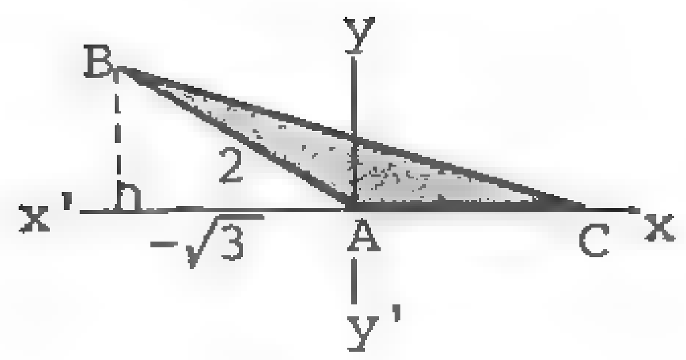
Solution: The domain of the sine function is the set of real numbers, and its range is the interval $\{-1 \le y \le 1\}$. For each number y in this interval, the equation $y = \sin x$ has infinitely many solutions. For example, for the number 0 in $\{-1 \le y \le 1\}$,

$$0 = \sin 0^\circ = \sin 360^\circ = \sin 720^\circ \ \ldots \ .$$

Recall that a function has an inverse if for each number y in the range of the function there is only one number x in the domain of the function such that $y = f(x)$. Thus, the sine function does not have an inverse.

• PROBLEM 3-5

In ΔABC, $A = \text{arc}\cos\left(-\frac{\sqrt{3}}{2}\right)$. What is the value of A expressed in radians?



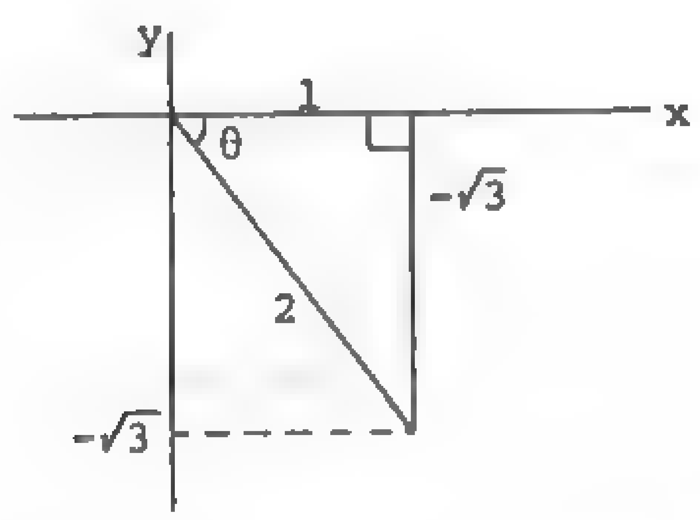
Solution: The expression "arc $\cos\left(-\frac{\sqrt{3}}{2}\right)$" means "the angle whose cosine equals $-\frac{\sqrt{3}}{2}$." Angles whose cosine equals $-\frac{\sqrt{3}}{2}$ are

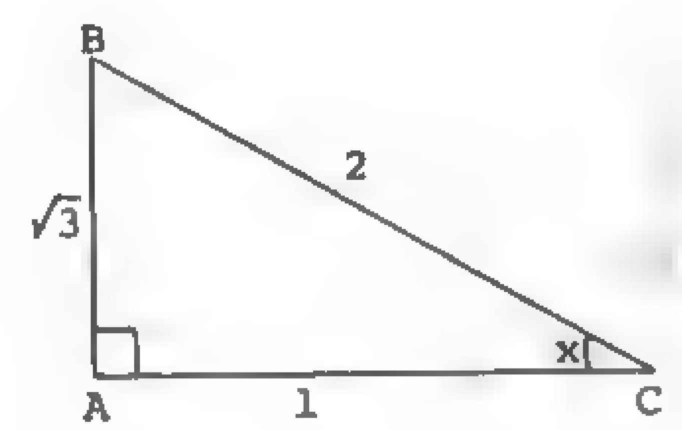
150°, 210°, -150°, and -210°.

Since the principal value of an arc cosine of an angle is the positive angle having the smallest numerical value of the angle, 150°, or $\frac{5\pi}{6}$, is the principal value of angle A.

• PROBLEM 3-6

Evaluate: (a) $\sin^{-1}\frac{\sqrt{3}}{2}$, (b) $\tan^{-1}\left(-\sqrt{3}\right)$.





Solution: (a) Recall that inverse sines are angles. Thus we are looking for the angle whose sin is $\frac{\sqrt{3}}{2}$. $\sin^{-1}\frac{\sqrt{3}}{2} = x$ means $\sin x = \frac{\sqrt{3}}{2}$ where $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$.

We note that triangle ABC is a 30-60 right triangle, and angle $x = 60^\circ$. Since $\sin 60^\circ = \frac{\sqrt{3}}{2}$,

$$\sin^{-1}\frac{\sqrt{3}}{2} = 60^\circ .$$

(b) Recall that inverse tangents are angles. Thus we are looking for the angle whose tangent is $-\sqrt{3}$. $\mathrm{Tan}^{-1}\left(-\sqrt{3}\right) = \theta$ means $\tan\theta = -\sqrt{3}$ where $\tan = \frac{\text{opposite}}{\text{adjacent}}$.

Since tangent is negative in the 4th quadrant, we draw our triangle there, and note it is a 30-60 right triangle, and angle $\theta = (-60^\circ)$.

Since $\tan(-60^\circ) = \frac{-\sqrt{3}}{1}$, $\tan^{-1}\left(-\sqrt{3}\right) = -60^\circ$.

• PROBLEM 3-7

Evaluate $\arctan\frac{5}{12} - \arccos\frac{3}{5}$.

Solution: Let $\alpha = \arctan\frac{5}{12}$, $\beta = \arccos\frac{3}{5}$. We are to evaluate the angle $\alpha - \beta$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $0 \le \beta \le \pi$. We first find some function of $\alpha - \beta$, say $\sin(\alpha - \beta)$. Since $\tan\alpha = \frac{5}{12}$, using the Pythagorean theorem and the fact that $\tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$, we obtain hypotenuse = 13; thus $\sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$. Also, since $\cos\beta = \frac{3}{5}$, $\sin\beta = \frac{4}{5}$. Then, using the formula for the sin of the difference of two angles, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{5}{13}\cdot\frac{3}{5} - \frac{12}{13}\cdot\frac{4}{5}$$
$$= \frac{15}{65} - \frac{48}{65} = -\frac{33}{65} = -.5077$$

approximately. Now, observe that α is less than $\frac{\pi}{4}$ and β is greater than $\frac{\pi}{4}$ (this can be seen by observing that $\sin\alpha = \frac{5}{13}$ and $\sin\beta = \frac{4}{5}$, and also observing the sin function from 0 to $\frac{\pi}{2}$). Hence $\alpha - \beta$ is a negative angle. Reference to tables gives us

$$\alpha - \beta = -\arcsin .5077 = -.5325 \text{ approximately.}$$

• PROBLEM 3-8

If principal values are used, then what does the ratio $\frac{\text{arc sin } \frac{1}{2}}{\text{arc tan } 1}$ equal?

Solution: The expression "arc sin $\frac{1}{2}$" means "the angle whose sine equals $\frac{1}{2}$"; similarly, "arc tan 1" means "the angle whose tangent equals 1."

Since the principal value of an arc sine or an arc tangent of an angle is the smallest numerical value of the angle,

$$\text{arc sin } \tfrac{1}{2} = 30^\circ \text{ and arc tan } 1 = 45^\circ.$$

Hence, the ratio

$$\frac{\text{arc sin } \frac{1}{2}}{\text{arc tan } 1} = \frac{30^\circ}{45^\circ} = \frac{2}{3} \quad .$$

• PROBLEM 3-9

Find $\sin^{-1} 0.4075$.

Solution: The inverse sine of 0.4075 is the angle, x, whose sine is 0.4075. Since 0.4075 is not found in the sine table, we proceed by finding the two numbers closest to .4075 (one greater and the other less than) which do appear in the table, and interpolating.

$$10'\left[d\left[\begin{array}{l} \sin 24^\circ 0' \\ \sin x \end{array}\right. \begin{array}{l} = 0.4067 \\ = 0.4075 \end{array}\right] 0.0008 \atop \sin 24^\circ 10' = 0.4094 \right] 0.0027$$

Setting up the proportion, $\frac{d}{10} = \frac{0.0008}{0.0027} = \frac{8}{27}$. Cross multiplying, $27d = 80$

$d = 3'$ to the nearest minute.

Therefore,

$$\sin^{-1} 0.4075 = 24^\circ 0' + 3' = 24^\circ 3'$$

Each function value is the function value of many angles. For example, a negative cosine is the cosine of an angle in the second quadrant and also of an angle in the third quadrant. But we defined the inverse relation so that it is a function. Hence, $\sin^{-1} 0.4075$ defines a unique angle, namely, 24°3'.

• PROBLEM 3-10

Find Arccos 0.74652.

Solution: The problem is to find the angle θ whose cosine function has value 0.74652. In a table of trigonometric functions, the number 0.74652 does not appear in the column headed by cos. We find two numbers, 0.74664 and 0.74644, one of which is larger and the other

smaller than 0.74652. In such a case, we must interpolate to obtain the proper angle. It must lie between 41°42' and 41°43'. The form for interpolation may be used.

	Angles	Values of Function		
1'	x [41°42'	0.74664		
	θ	0.74652	−0.00012	−0.00020
	41°43'	0.74644		

Note that the negative values found in the values of the cosine function were obtained by subtracting a number from the one below it. The same was done for the angles. Illustrating this point: 41°43' − 41°42' = 1', θ − 41°42' = x, 0.74652 − 0.74664 = −0.00012, and 0.74644 − 0.74664 = −0.00020.

Now the following proportion is set up:

$$\frac{x}{1'} = \frac{-0.00012}{-0.00020}$$

$$\text{or } x = \frac{-0.00012}{-0.00020}(1') = \frac{-0.00012}{-0.00020}(60")$$

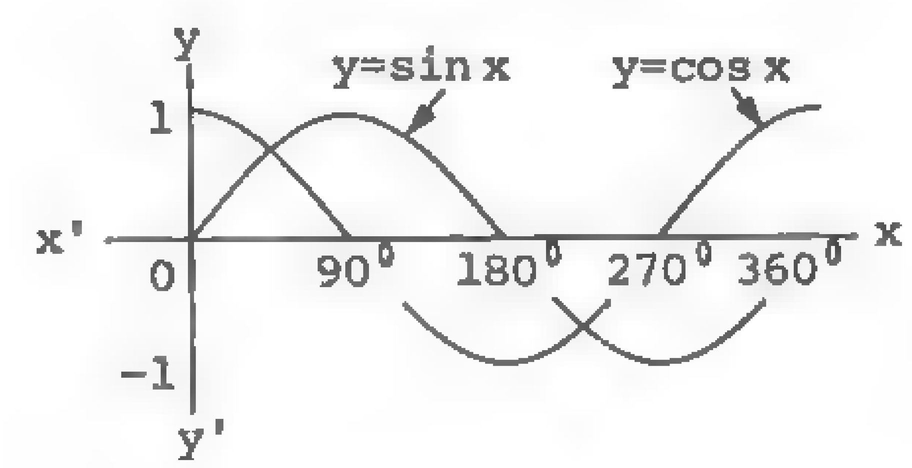
$$= (.6)(60") = 36.0" = 36"$$

Therefore, x = 41°42' 0" + 0° 0'36" = 41°42'36".

Hence, Arccos 0.74652 = θ = 41°42'36".

• PROBLEM 3-11

Evaluate cos[arc sin (-1)].



Solution: The expression "arc sin (-1)" means "the angle whose sine equals -1." Between 0° and 360°, the only angle whose sine equals -1 is 270°.

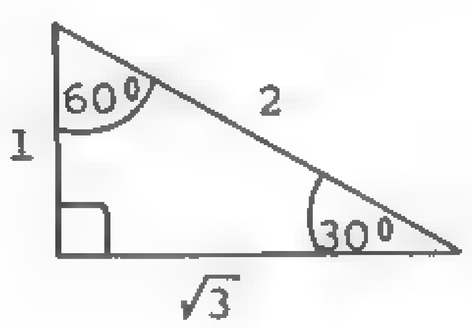
Hence, $\cos[\text{arc}\sin(-1)] = \cos 270^\circ$.

The value of $\cos 270^\circ = 0$.

Note: In problems of this type, a sketch of $y = \sin x$ and $y = \cos x$ is very useful.

• PROBLEM 3-12

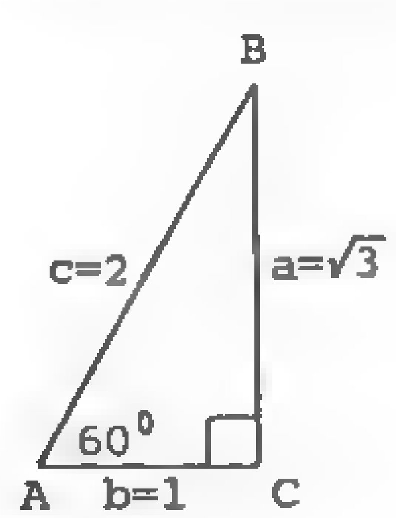
Evaluate $\cos\left(\sin^{-1}\frac{1}{2}\right)$.



Solution: Inverse sine of $\frac{1}{2}$ is the angle whose sin is $\frac{1}{2}$. From our diagram of a $30° - 60° - 90°$ right triangle we observe $\sin 30° = \frac{1}{2}$ so $\sin^{-1}\frac{1}{2} = 30°$. Thus $\cos\left(\sin^{-1}\frac{1}{2}\right) = \cos(30°)$. Consulting our diagram we see $\cos 30° = \frac{\sqrt{3}}{2}$. Therefore $\cos\left(\sin^{-1}\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$.

• PROBLEM 3-13

Express in radical form the positive value of $\sin(\text{arc}\cos\frac{1}{2})$.

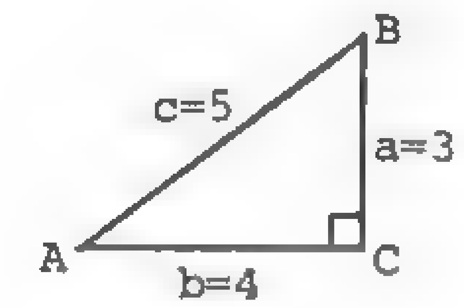


Solution: The expression "arc cos $\frac{1}{2}$" means "the angle whose cosine equals $\frac{1}{2}$." In the diagram, note that A is the angle whose cosine is $\frac{1}{2}$.

Since 60° is the angle whose cosine equals $\frac{1}{2}$, $A = 60^\circ$. The positive value of $\sin(\text{arc}\cos\frac{1}{2}) = \sin 60^\circ = \frac{\sqrt{3}}{2}$.

• PROBLEM 3-14

Find $\sin\left(\text{Arc}\tan\frac{3}{4}\right)$.



Solution: The expression "arc tan $\frac{3}{4}$" means "the angle whose tangent equals $\frac{3}{4}$." Spelled with a capital "A", "Arc tan" refers to the principal value of that angle, that is, its measure as a positive acute angle.

Draw a right triangle ABC with an acute angle A such that the leg adjacent to A is 4 and the leg opposite A is 3. Thus, $\tan A = \frac{3}{4}$ and the expression "Arc tan $\frac{3}{4}$" can be replaced by A.

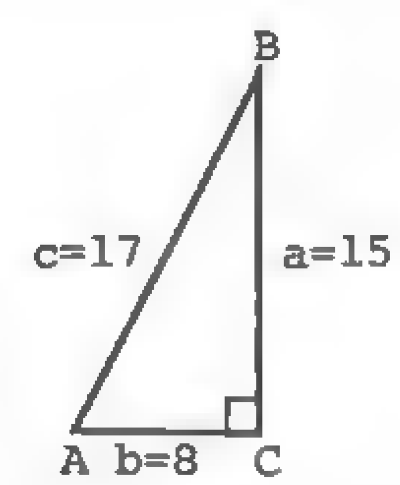
Hence, $\sin\left(\text{Arc}\tan\frac{3}{4}\right) = \sin A$.

Apply the Law of Pythagoras: if the legs of a right triangle are 3 and 4, its hypotenuse is 5. Hence, $c = 5$.

Therefore, $\sin A = \frac{3}{5}$, or $\sin\left(\text{Arc}\tan\frac{3}{4}\right) = \frac{3}{5}$.

• PROBLEM 3-15

Find the value of $\sin\left(\text{Arc}\cos\frac{8}{17}\right)$.



Solution: The expression "arc cos $\frac{8}{17}$" means "the angle whose cosine equals $\frac{8}{17}$." Spelled with a capital "A", "Arc cos" refers to the principal value of that angle, that is, its measure as a positive acute angle.

Draw a right ΔABC with an acute angle A such that the leg adjacent to A is 8 and the hypotenuse 17. Thus, $\cos A = \frac{8}{17}$, and the

expression "Arc cos $\frac{8}{17}$" can be replaced by A.

Hence, $\sin\left(\text{Arc cos } \frac{8}{17}\right) = \sin A = \frac{a}{c}$.

Since $c = 17$ and $b = 8$, then $a^2 = 17^2 - 8^2$

$$= 289 - 64 = 225$$

Hence, $a = \sqrt{225} = 15$

Therefore, $\frac{a}{c} = \frac{15}{17}$ or

$$\sin\left(\text{Arc cos } \frac{8}{17}\right) = \frac{15}{17} .$$

• PROBLEM 3-16

> What is the value of $\tan\left(\text{Arc cos } \frac{\sqrt{2}}{2}\right)$?

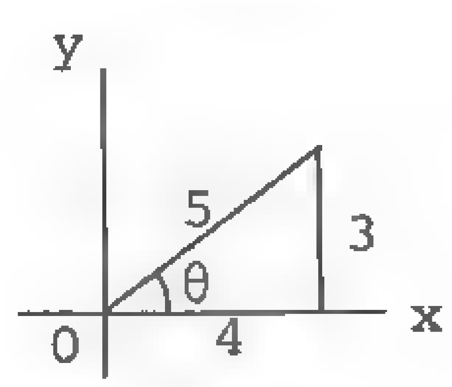
Solution: The expression "arc cos $\frac{\sqrt{2}}{2}$ " means "the angle whose cosine is $\frac{\sqrt{2}}{2}$ " . Spelled with a capital "A," "Arc cos" refers to the principle value of that angle. If the cosine is positive, the principle value of an arc cosine is its measure as a positive acute angle.

Hence, Arc cos $\frac{\sqrt{2}}{2} = 45^\circ$.

Therefore, $\tan\left(\text{Arc cos } \frac{\sqrt{2}}{2}\right) = \tan 45^\circ = 1.$

• PROBLEM 3-17

> Find sin arccos $\frac{4}{5}$, if arccos $\frac{4}{5}$ is in quadrant I.



Solution: Let $\theta = \arccos \frac{4}{5}$; then $\cos \theta = \frac{4}{5}$. We can then construct the triangle of the figure. From the triangle we get $\sin \theta = \frac{3}{5}$; therefore $\sin \arccos \frac{4}{5} = \frac{3}{5}$.

• PROBLEM 3-18

> Calculate $\sin\left(\cos^{-1} \frac{3}{5}\right)$.

Solution: Let $t = \cos^{-1} \frac{3}{5}$. This means that $\cos t = \frac{3}{5}$. We restrict t to $0 \le t \le \pi$ to obtain the inverse of the cos function also as a function. But since $\cos t > 0$ we know that $0 \le t < \frac{1}{2}\pi$. Now, $\cos^2 t = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$. We are looking for $\sin t$, and by use of the identity, $\sin^2 t + \cos^2 t = 1$, we have:

$$\sin^2 t + \frac{9}{25} = 1$$
$$\sin^2 t = 1 - \frac{9}{25} = \frac{16}{25} .$$

Since $0 \le t < \frac{1}{2}\pi$, we know that $\sin t \ge 0$, and thus $\sin t = \frac{4}{5}$.
Hence,

$$\left(\cos^{-1} \frac{3}{5}\right) = \frac{4}{5} .$$

• PROBLEM 3-19

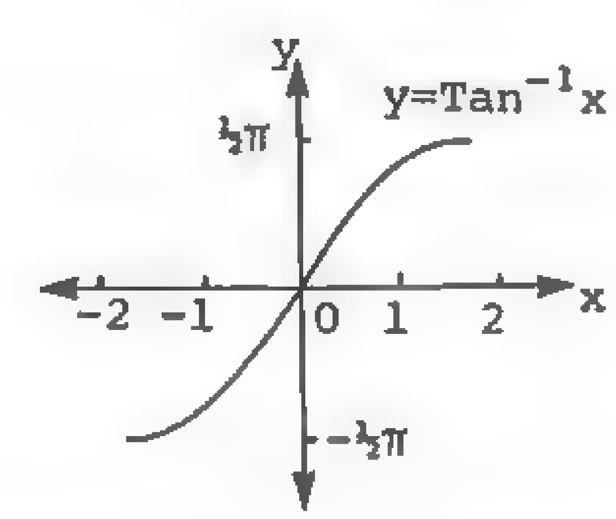
> Find $\sin\left(\frac{1}{2} \text{Arccos } \frac{5}{15}\right)$.

Solution: Let Arccos $\frac{5}{13} = \alpha$; then we have

$$\sin\left(\frac{1}{2}\alpha\right) = \sqrt{\frac{1 - \cos\alpha}{2}} = \sqrt{\frac{1 - \frac{5}{13}}{2}} = \sqrt{\frac{8}{26}} = \frac{2}{13}\sqrt{13} .$$

• PROBLEM 3-20

> Find sin(Arctan x), where x may be any real number.



Solution: Let $t = \text{Arctan } x$. Then $\tan t = x$, and $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$. Also t and x have the same sign (see Figure). Now, we express $\sin t$ in terms of x as follows:

since $\tan t = x = \dfrac{\sin t}{\cos t}$, and

$$\sin t = \frac{\sin t}{\cos t} \cdot \cos t ,$$

we write

$$\sin t = x \cos t$$

$$\sin^2 t = x^2 \cos^2 t,$$

and by the identity

$$\sin^2 t + \cos^2 t = 1,$$

$$\sin^2 t = x^2 (1 - \sin^2 t).$$

Solving for $\sin^2 t$,

$$1 = \frac{x^2 (1 - \sin^2 t)}{\sin^2 t}$$

$$1 = \frac{x^2 - x^2 \sin^2 t}{\sin^2 t}$$

$$1 = \frac{x^2}{\sin^2 t} - \frac{x^2 \sin^2 t}{\sin^2 t}$$

$$1 = \frac{x^2}{\sin^2 t} - x^2$$

$$1 + x^2 = \frac{x^2}{\sin^2 t}$$

$$\sin^2 t(1 + x^2) = x^2$$

$$\sin^2 t = \frac{x^2}{1 + x^2} .$$

Hence,

$$\sin t = \frac{x}{\sqrt{1 + x^2}} \quad \text{or} \quad \frac{-x}{\sqrt{1 + x^2}} .$$

Since $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$, $\sin t$ and t have the same sign. From above we know that t and x have the same sign. Thus, $\sin t$ and x must have the same sign. Now

$$\frac{x}{\sqrt{1 + x^2}}$$

has the same sign as x, while

$$\frac{-x}{\sqrt{1 + x^2}}$$

has the opposite sign; thus $\sin t = \sin(\text{Arctan } x) = \dfrac{x}{\sqrt{1 + x^2}}$.

• PROBLEM 3-21

Find the positive value of $\tan\left(\text{arc sin } \frac{\sqrt{2}}{2}\right)$.

Solution: The expression within the parentheses, "arc sin $\frac{\sqrt{2}}{2}$," means "the angle whose sine equals $\frac{\sqrt{2}}{2}$." Angles whose sine equals $\frac{\sqrt{2}}{2}$ are 45°, 135°, -225°, -315°, and other angles that can be obtained from these by adding or subtracting 360° or a multiple of 360°.

Hence, the expression "$\tan\left(\text{arc sin } \frac{\sqrt{2}}{2}\right)$" may mean $\tan 45^\circ$, $\tan 135^\circ$, or the tangent of any of the other angles. The value of these tangents is either $+1$ or -1. However, only the positive value is required, thus $\tan\left(\text{arc sin } \frac{\sqrt{2}}{2}\right) = 1$.

• PROBLEM 3-22

Evaluate $\tan\left[\frac{1}{2} \text{ arcsin } \left(-\frac{8}{17}\right)\right]$.

Solution: Let $\theta = \text{arcsin } \left(-\frac{8}{17}\right)$. Then $\sin \theta = -\frac{8}{17}$. Thus, we wish to evaluate $\tan\left(\frac{\theta}{2}\right)$. Using the half-angle formula for tan we obtain:

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{\sin \theta} .$$

Thus, we must find, in addition to $\sin \theta = -\frac{8}{17}$, the corresponding value of $\cos \theta$. Observe that when $\sin \theta$ is negative, θ must be a negative angle in the fourth quadrant, since we have the restriction on the inverse sin function $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Thus, $\cos \theta$ will be positive (since cos is positive in the fourth quadrant), and using the identity $\sin^2\theta + \cos^2\theta = 1$ we obtain:

$$\cos \theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(-\frac{8}{17}\right)^2} = \sqrt{1 - \frac{64}{289}}$$

$$= \sqrt{\frac{225}{289}} = \frac{15}{17} .$$

Therefore the desired value is

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \frac{15}{17}}{-\frac{8}{17}} = \frac{2}{17} \cdot \left(-\frac{17}{8}\right) = -\frac{1}{4} ,$$

or

$$\tan\left[\frac{1}{2} \text{ arcsin } \left(-\frac{8}{17}\right)\right] = -\frac{1}{4} .$$

• **PROBLEM 3-23**

Find cos (Arccos x - Arccos 3y).

Fig. 1

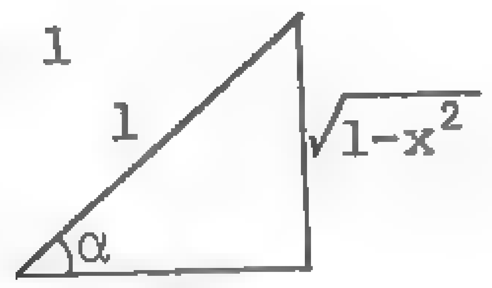
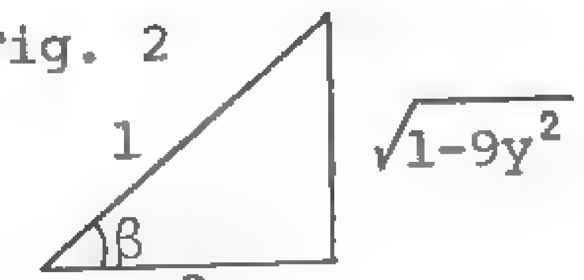


Fig. 2



Solution: Let Arccos $x = \alpha$ and Arccos $3y = \beta$. We can then construct the triangles of Figs. 1 and 2.

$$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= (x)(3y) + (\sqrt{1-x^2})(\sqrt{1-9y^2})\\ &= 3xy + \sqrt{(1-x^2)(1-9y^2)}\ .\end{aligned}$$

Note: $\sin\alpha$ and $\sin\beta$ are both positive since α and β must be between 0 and π; we thus take only the positive radical.

• **PROBLEM 3-24**

Find sin $(\sin^{-1} 1 + \sin^{-1} \frac{1}{2})$.

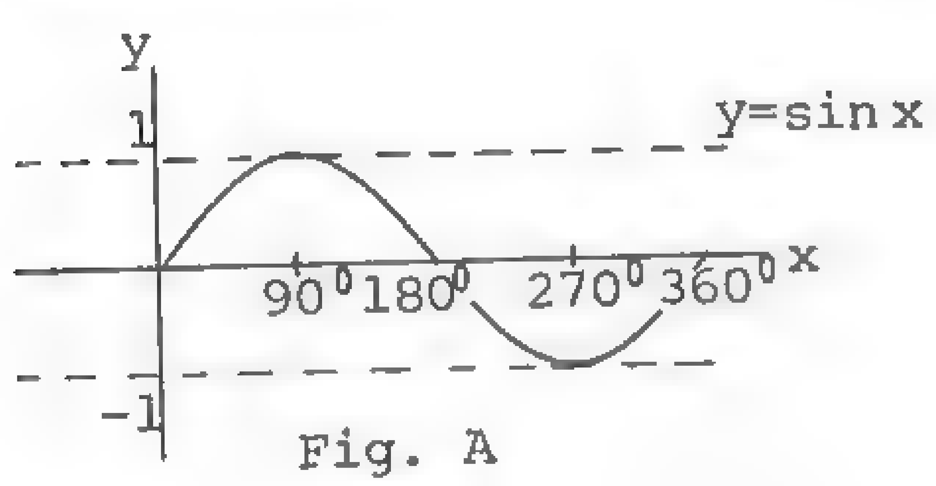


Fig. A

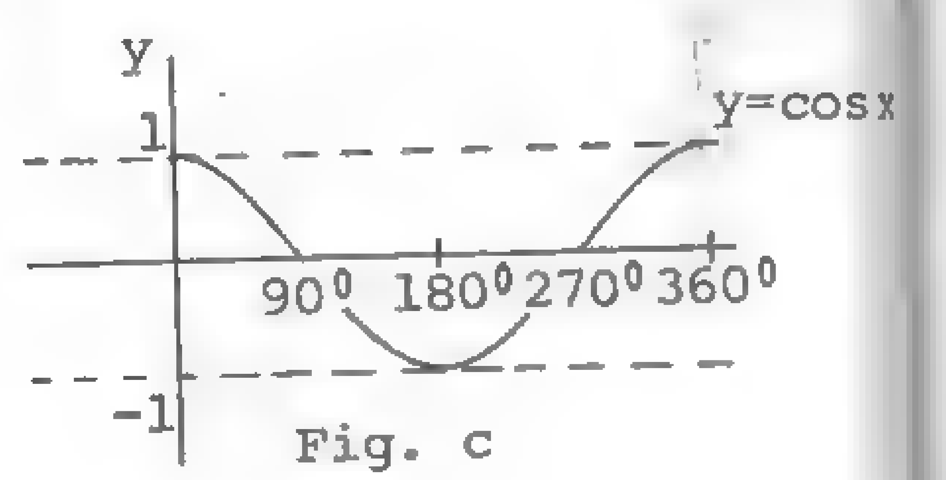


Fig. c

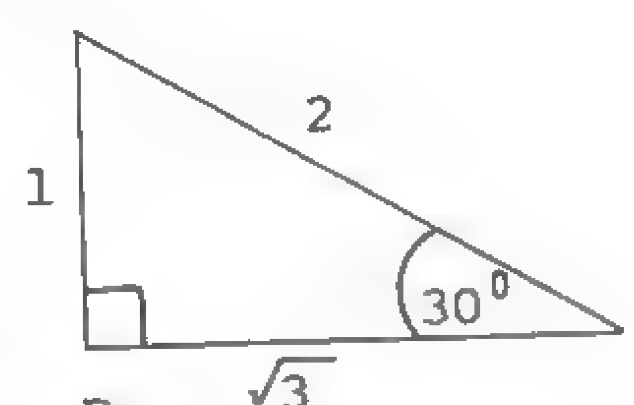


Fig. B

Solution: Recall that inverse sines are angles. Hence $\sin^{-1} 1$ is the angle whose sine is 1.

From the diagram of the sine function, we see that the angle whose sine is 1 is 90°.

Similarly, $\sin^{-1}\frac{1}{2}$ is the angle whose sine is $\frac{1}{2}$; that is, the angle whose opposite side is 1 and whose hypotenuse is 2.

We note this is a 30-60 right triangle and the angle whose sine is $\frac{1}{2}$ is a 30° angle hence,

$$\sin(\sin^{-1} 1 + \sin^{-1}\tfrac{1}{2}) = \sin(90° + 30°)$$

Using the sum of the sines formula

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y:\\ &= \sin 90° \cos 30° + \cos 90° \sin 30°\end{aligned}$$

Now we must find the values of sin 90°, cos 30°, cos 90°, and sin 30°.

We observed from our sine graph, sin 90° = 1. To calculate cos 90° we can observe the graph of the cos function.

Thus we see cos 90° = 0.

To calculate cos 30° and sin 30° we look at a 30-60 right triangle:

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}\ ,\ \text{hence}\ \cos 30° = \frac{\sqrt{3}}{2}$$

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}\ ,\ \text{hence}\ \sin 30° = \tfrac{1}{2}$$

$$= 1\cdot\frac{\sqrt{3}}{2} + 0\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}\ .$$

• **PROBLEM 3-25**

Show that if $x > 0$, then Arctan x = Arccot $\frac{1}{x}$.

Solution: Let u = Arctan x. Then tan u = x, and $-\frac{1}{2}\pi < u < \frac{1}{2}\pi$. Since $x > 0$, $0 < u < \frac{1}{2}\pi$. Observe that

$$\frac{1}{x} = \frac{1}{\tan u} = \cot u\ ,$$

and $0 < u < \frac{1}{2}\pi$. Since $\frac{1}{x} = \cot u$ and $0 < u < \frac{1}{2}\pi$ we have u = Arccot$\left(\frac{1}{x}\right)$. But we have already stated that u = Arctan x. Hence Arctan x = Arccot $\frac{1}{x}$.

• **PROBLEM 3-26**

Show that Arcsin x + Arccos x = $\frac{1}{2}\pi$ for any number x such that $-1 \le x \le 1$.

Solution: Let $u = \text{Arcsin } x$. Then $\sin u = x$. We restrict u such that $-\frac{1}{2}\pi \le u \le \frac{1}{2}\pi$ so that the inverse of the sin function is also a function. Now let $v = \frac{1}{2}\pi - u$; then $0 \le v \le \pi$, and $\cos v = \cos(\frac{1}{2}\pi - u)$. Recall the formula for the cosine of the difference of two angles, $\cos(a - b) = \cos a \cos b + \sin a \sin b$. Using this we obtain:

$$\cos(\tfrac{1}{2}\pi - u) = \cos \tfrac{1}{2}\pi \cos u + \sin \tfrac{1}{2}\pi \sin u,$$

and since $\cos \frac{1}{2}\pi = 0$ and $\sin \frac{1}{2}\pi = 1$, we have:

$$\cos(\tfrac{1}{2}\pi - u) = \sin u = x.$$

Since $0 \le v \le \pi$, and $\cos v = x$, we have $v = \text{Arccos } x$. Hence,

$$\text{Arcsin } x + \text{Arccos } x = u + v = u + (\tfrac{1}{2}\pi - u) = \tfrac{1}{2}\pi .$$

• PROBLEM 3-27

Solve the equation $\arcsin x + \arccos(1 - x) = 0$.

Solution: Let $\alpha = \arcsin x$, $\beta = \arccos(1 - x)$. Then we must solve the equation

$$\alpha + \beta = 0,$$

where $\sin \alpha = x$ and $\cos \beta = 1 - x$. Now, by use of the identity $\sin^2 a + \cos^2 a = 1$ we obtain:

$$\cos \alpha = \pm\sqrt{1 - \sin^2\alpha} = \pm\sqrt{1 - x^2} \quad ,$$

and

$$\sin \beta = \pm\sqrt{1 - \cos^2\beta} = \pm\sqrt{1 - (1-x)^2} = \pm\sqrt{2x - x^2} \ .$$

We now make use of the formula for the sin of the sum of two angles which states,

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \ ;$$

and this equals 0 since $\alpha + \beta = 0$, and $\sin 0 = 0$. Substituting the above values for $\sin \alpha$, $\cos \beta$, $\cos \alpha$, $\sin \beta$ we have:

$$\sin \alpha \cos \beta + \cos \alpha \sin \beta = 0$$

$$x(1 - x) \pm \sqrt{1 - x^2}\sqrt{2x - x^2} = 0$$

$$x^2(1 - x)^2 = (1 - x^2)(2x - x^2)$$

$$x^2(1 - 2x + x^2) = (1 - x^2)(2x - x^2)$$

Observe that $x = 0$ satisfies this equation. Substituting this value in the given equation we obtain:

$$\arcsin 0 + \arccos(1 - 0) = 0.$$

Now, since $\arcsin 0 = 0$ and $\arccos 1 = 0$ we have:

$$0 + 0 = 0;$$
$$0 = 0$$

thus this value of x also satisfies the original equation. Removing the factor x from $x^2(1 - 2x + x^2) = (1 - x^2)(2x - x^2)$ yields:

$$x^2(1 - 2x + x^2) = (1 - x^2)(2 - x)x$$

$$x(1 - 2x + x^2) = (1 - x^2)(2 - x)$$

$$x - 2x^2 + x^3 = 2 - x - 2x^2 + x^3$$

$$2x = 2$$
$$x = 1$$

Thus, $x = 1$ is another possible solution to the original equation. But $\arcsin 1 = \pi/2$ and $\arccos 0 = \pi/2$, thus

$$\arcsin 1 + \arccos(1 - 1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi \ne 0.$$

Thus, $x = 1$ does not satisfy the given equation. (Notice that outside the restricted values for arccos we have $\arccos 0 = -\pi/2$ also, which, together with $\arcsin 1 = \pi/2$, makes $x = 1$ a solution.)

Hence $x = 0$ is the only solution of the given equation when only the restricted values are permitted (recall that for the inverse sine function,

$$-\frac{\pi}{2} \le (\arcsin x = \alpha) \le \frac{\pi}{2}$$

and for the inverse cos function

$$0 \le (\arccos(1 - x) = \beta) \le \pi).$$

CHAPTER 4

TRIGONOMETRIC FUNCTIONS

• PROBLEM 4-1

Find the solution set for $\sin x = 0$.

Solution: The solution set consists of all distinct values of the angle x which satisfy the equation. Note that $x = 0$ and $x = \pi$ are both elements of the solution set. Similarly, any angle $n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, satisfies the equality. Thus, the solution set is $\{x \mid x = n\pi, n = 0, \pm 1, \pm 2, \ldots\}$. This set is an infinite set.

• PROBLEM 4-2

Find the solution set of $\cos x = 3$.

Solution: Since the values of the cosine function only range from -1 to $+1$; that is, since $-1 \le \cos x \le 1$, there is no value of x which satisfies the equality $\cos x = 3$. Therefore, the solution set is the empty set.

• PROBLEM 4-3

Find the solution set on $[0,2\pi)$ for $\sin x = \cos x$.

Solution: Since the equation does not lend itself to factoring, we divide both sides by $\cos x$, obtaining $\sin x/\cos x = 1$, or $\tan x = 1$. The solution set of this new equation is $\{\pi/4, 5\pi/4\}$. However, in dividing the original equation by $\cos x$, it was assumed that

$\cos x \neq 0$ because division by 0 is not permitted. However, $\cos x$ may be equal to zero. When $\cos x = 0$, $x = \pi/2$ or $x = 3\pi/2$. Checking the four solutions $x = \pi/4, \pi/2, 5\pi/4, 3\pi/2$ in the original equation:

$$\sin \pi/4 = 0.7071 = \cos \pi/4 \checkmark$$

$$\sin \pi/2 = 1 \neq \cos \pi/2 \text{ since } \cos \pi/2 = 0.$$

$$\sin 5\pi/4 = -0.7071 = \cos 5\pi/4 \checkmark$$

$$\sin 3\pi/2 = -1 \neq \cos 3\pi/2 \text{ since } \cos 3\pi/2 = 0.$$

Therefore, the solution set of the original equation is:

$$\{\pi/4, 5\pi/4\}.$$

• PROBLEM 4-4

Find the solution set on $[0,2\pi]$ of the equation $\sqrt{1 + \sin^2 x} = \sqrt{2} \sin x$.

Solution: Since the unknown quantity is involved in the radicand, squaring of both sides to eliminate the radical is suggested. Thus, we obtain $1 + \sin^2 x = 2 \sin^2 x$. Hence, $\sin^2 x - 1 = 0$,

or $\sin^2 x = 1$

$$\sqrt{\sin^2 x} = \pm\sqrt{1}$$

$$\sin x = \pm 1$$

When $\sin x = 1$ on $[0,2\pi]$, $x = \pi/2$. When $\sin x = -1$ on $[0,2\pi]$, $x = 3\pi/2$.

The complete solution set seems to be $\{\pi/2, 3\pi/2\}$. Since we squared both sides of the equation, we should try each element in the original equation. When $x = \pi/2$, we obtain $\sqrt{1+1} = \sqrt{2}\cdot 1$. When $x = 3\pi/2$, we obtain $\sqrt{1+1} = \sqrt{2}(-1)$. The second element does not satisfy the original equation, hence does not belong to the solution set. An extraneous root was introduced by squaring the equation, it would seem. Thus, the solution set is $\{\pi/2\}$.

• PROBLEM 4-5

Find one number in the solution set of the equation $\tan t = \sqrt{3}$.

Fig. A

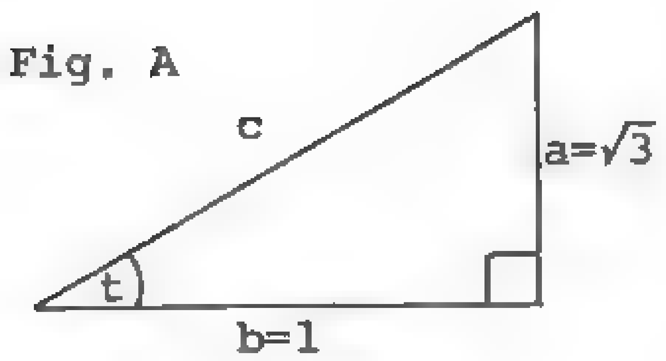
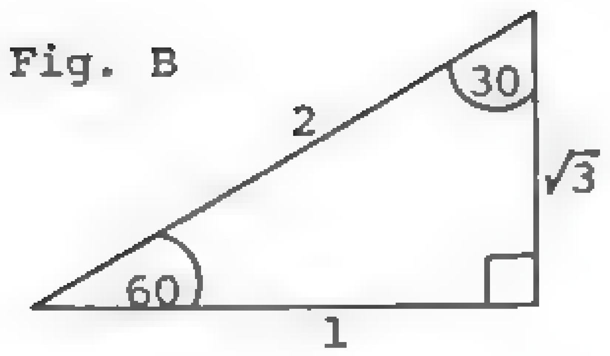


Fig. B



Solution: We know that the

$$\tan t = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sqrt{3}}{1} \quad \text{(see figure A).}$$

The third side, c, can be found by the Pythagorean theorem $c^2 = a^2 + b^2$

$$c^2 = \left(\sqrt{3}\right)^2 + 1^2 = 3 + 1 = 4$$

$$c = \sqrt{4} = \pm 2$$

Reject -2, since a side of a triangle cannot be negative. Thus, $c = +2$. The student should recall the trigonometric functions of the 30°, 60°, 90° triangle, that is,

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \tfrac{1}{2} = \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{(see figure B).}$$

Thus, $t = 60^\circ$ because $\cos t = \frac{1}{2}$, and $\sin t = \frac{\sqrt{3}}{2}$; that is,

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\sqrt{3}/2}{1/2} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{1}\right) = \sqrt{3}$$

• PROBLEM 4-6

Find the solution set on $[0, \pi]$ for the equation
$$\tan x \sin x - \sin x - \tan x + 1 = 0.$$

Solution: This equation can be factored to obtain
$$(\sin x - 1)(\tan x - 1) = 0.$$
The values of x satisfying this equation may be found by setting each factor equal to zero.

$$\sin x - 1 = 0\ ; \quad \tan x - 1 = 0$$

or

$$\sin x = 1\ ; \quad \tan x = 1$$

Keeping in mind that our solution set cannot contain values exceeding π or less than zero. We find that $x = \pi/2$, is our only acceptable solution for the first equation and $x = \pi/4$ is the only acceptable solution for the second, i.e.,

$\sin(k \cdot \pi/2) = 1$ where $k = 1,3,5,\ldots$ but $k = 3,5,\ldots$ is unacceptable

$$\tan(k\pi/4) = \frac{\sin(k \cdot \pi/4)}{\cos(k \cdot \pi/4)} = \frac{\sqrt{2}/2}{\sqrt{2}/2}, k = 1,3,5,\ldots \quad \text{but}$$

$k = 3,5,\ldots$ is also unacceptable.

By substituting $\pi/2$ into $(\sin x - 1)(\tan x - 1) = 0$ we arrive at the undefined quantity $0x(1/0 - 1)$ so we must specify that $\pi/4$ is our only solution as its substitution leads to a valid identity.

• PROBLEM 4-7

Find all angles on $[0^\circ, 360^\circ)$ which satisfy $\sin 2x - \sqrt{2} \sin x = 0$.

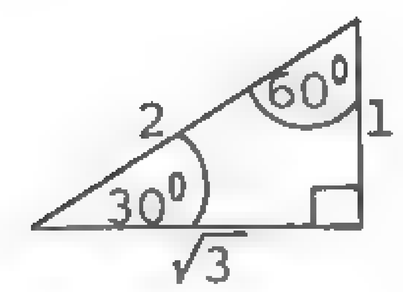


Fig. A

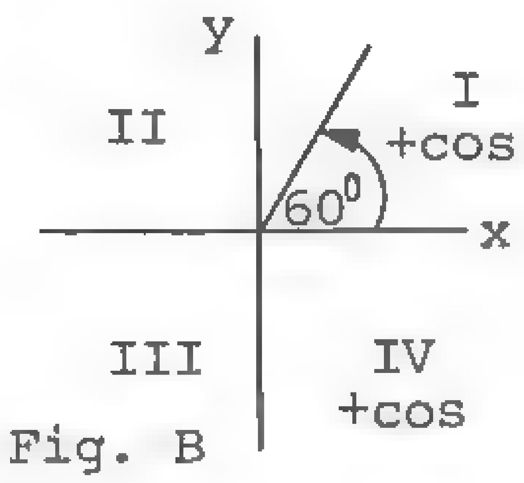


Fig. B

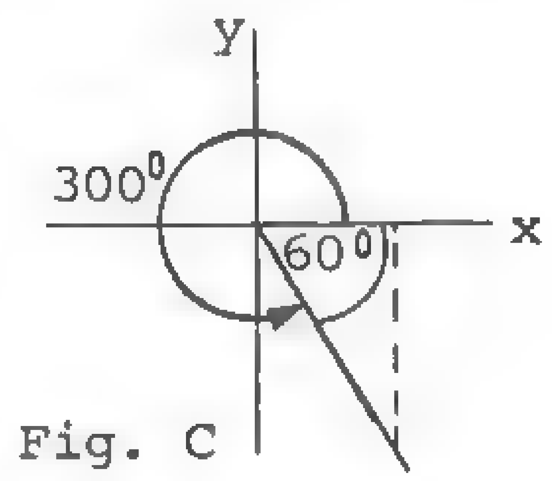


Fig. C

Solution: The equation contains mixed expressions and must be expressed as an equation involving only one multiple of the angle. It seems to be practical to express $\sin 2x$ in terms of the angle x; by using the double angle formula we can express $\sin 2x$ in terms of an angle multiplied by unity.

$$\sin(x + x) = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x$$

$$\sin 2x - \sqrt{2} \sin x = 2 \sin x \cos x - \sqrt{2} \sin x = 0$$

$$\sin x(2 \cos x - \sqrt{2}) = 0$$

Setting each factor = 0,

$$\sin x = 0$$
$$x = 0, \pi$$
$$2 \cos x - \sqrt{2} = 0$$
$$\cos x = \frac{\sqrt{2}}{2}$$
$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

The solution set is $\left\{0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}\right\}$

• PROBLEM 4-8

Solve $2 \sin^2\theta + 3 \cos \theta - 3 = 0$ for θ if $0 \le \theta < 360^\circ$.

Solution: The solution to the equation can be found by expressing the equation in terms of one trigonometric function. Here the convenient function is $\cos \theta$. Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can eliminate $\sin^2\theta$ from the equation by substituting $1 - \cos^2\theta$:

$$2\left(1 - \cos^2\theta\right) + 3 \cos \theta - 3 = 0$$

distributing: $2 - 2 \cos^2\theta + 3 \cos \theta - 3 = 0$

adding: $-2\cos^2\theta + 3 \cos \theta - 1 = 0$

multiply by -1: $2 \cos^2\theta - 3 \cos \theta + 1 = 0$

factoring: $(2 \cos \theta - 1)(\cos \theta - 1) = 0$

Hence, $\cos \theta = \frac{1}{2}$ or $\cos \theta = 1$

Observe Figure A, of a 30-60 right triangle. From it we find

$\cos 60^\circ = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{1}{2}$.

The cosine is positive in quadrants I and IV; thus, θ can be 60° or 300°. (See Figures B and C). Then,

$$\theta = 60^\circ, 300^\circ \quad \text{or} \quad \theta = 0^\circ (\text{since } \cos 0^\circ = 1)$$

Substitution verifies all three. The solution set is

$$\{0^\circ, 60^\circ, 300^\circ\}$$

By removing the restriction that $0 \le \theta < 360^\circ$, the solution set is $\{0^\circ + 360^\circ k,\ 60^\circ + 360^\circ k,\ 300^\circ k\}$, k an integer.

• PROBLEM 4-9

Find the solution set of $\sin^2 \theta + \sin \theta = 0$.

Solution: Factoring the left side of the equation, obtain $\sin \theta (\sin \theta + 1) = 0$. Setting each factor equal to zero and solving for $\sin \theta$, obtain $\sin \theta = 0$ and $\sin \theta = -1$. For $\sin \theta = 0$, $\theta = 0, \pi, \ldots.$ and all integral multiples of π.

For $\sin \theta = -1$, note that the sign is negative and that the value of the sine is one. Thus, $\theta = 3\pi/2$ and all integral multiples of 2π plus $3\pi/2$. Therefore, from the first equation, the solution set contains the elements $n\pi$, $n = 0, \pm 1, \ldots.$ From the second, the solution set contains the elements $3\pi/2 + 2n\pi$, $n = 0, \pm 1, \ldots$ The solution set of the original equation is then

$$\{\theta \mid \theta = n\pi, \text{ or } 3\pi/2 + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots\}$$

• PROBLEM 4-10

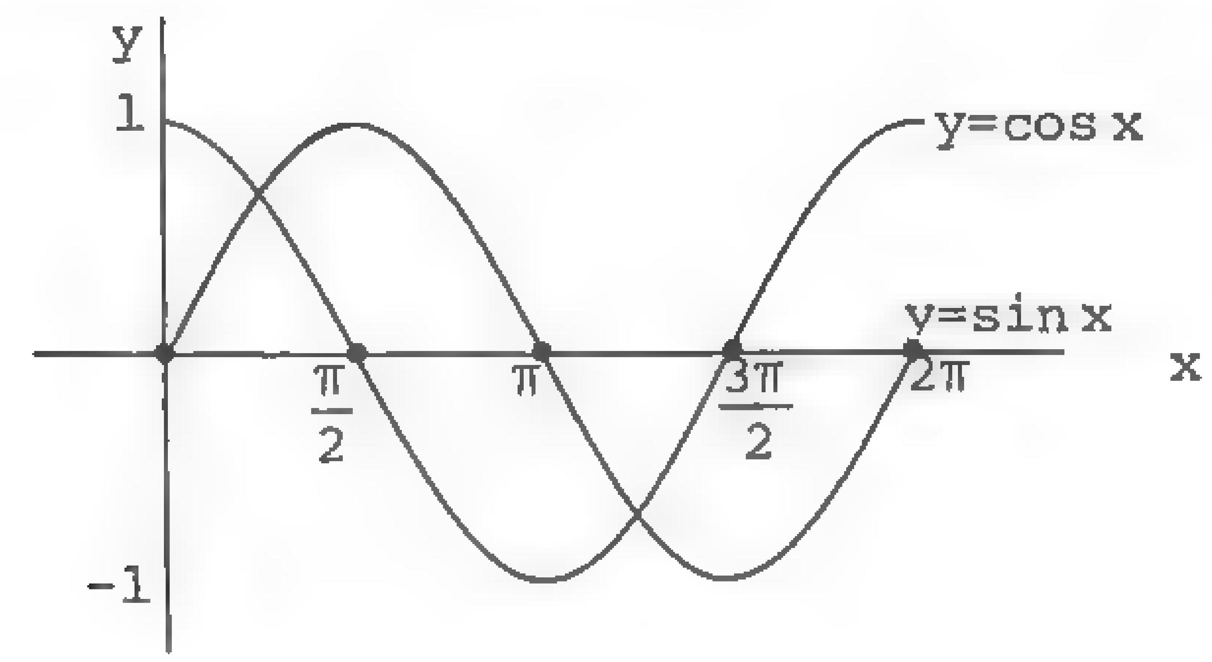
Find the solution set on $[0, 2\pi]$ for the equation $\sin x \cos x = \cos x$.

Solution: Dividing by $\cos x$ we obtain $\sin x = 1$. However, this operation assumes that $\cos x \neq 0$, since division by 0 is not permitted. Hence we have that $\cos x$ may also be equal to zero. Therefore, the solution set consists of all values of x which satisfy the two equations $\sin x = 1$ and $\cos x = 0$. When $\sin x = 1$ on $[0, 2\pi]$ $x = \pi/2$. When $\cos x = 0$ on $[0, 2\pi]$, $x = \pi/2$ or $x = 3\pi/2$.

Note that one of the values of x obtained from the cosine function ($x = \pi/2$) is the same as the value of x obtained from the sine function. Therefore, the complete solution set is $\{\pi/2, 3\pi/2\}$.

• PROBLEM 4-11

Solve for θ : $\sin \theta + 2 \tan \theta = 0$, $0 \le \theta \le 2\pi$.



Solution: $\sin \theta + 2 \tan \theta = 0$

Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, by substitution: $\sin \theta + 2 \frac{\sin \theta}{\cos \theta} = 0$

Multiplying both sides by $\cos \theta$ gives us $\cos \theta \left(\sin \theta + 2 \frac{\sin \theta}{\cos \theta} \right) = 0(\cos \theta)$

Distributing we obtain,

$$\sin \theta \cos \theta + 2 \sin \theta = 0$$

Factoring out $\sin \theta$ gives us

$$\sin \theta (\cos \theta + 2) = 0$$

If $xy = 0$, either $x = 0$ or $y = 0$, hence either $\sin \theta = 0$ or $\cos \theta + 2 = 0$. Subtracting 2 from each side of the latter gives us $\cos \theta = -2$ Thus $\sin \theta = 0$ or $\cos \theta = -2$

On the given interval, $0 \le \theta \le 2\pi$, $\sin\theta = 0$ when $\theta = 0$ or $\theta = \pi$; and $\cos \theta = -2$ for no angles of θ (cos is only defined on the interval $|-1, 1|$).

Thus the solution set is $\{0, \pi\}$. Check these values. If we do not restrict θ as we have done, then the solutions may be expressed as $\theta = 0 + k\pi$, where k is any integer. That is, by adding any integral multiple of π to the angle θ, we obtain an angle coterminal with either zero or π. (By coterminal we mean the initial sides of the angles lie on the positive branch of the x-axis and the terminal sides coincide.)

• **PROBLEM 4-12**

Solve the equation

$$\sin^2 \theta + 2 \cos \theta - 1 = 0$$

for non-negative values of θ less than 2π.

Solution: Two trigonometric functions of the unknown θ itself appear in this equation. Accordingly, we make use of the identity connecting these functions, namely, $\sin^2 \theta + \cos^2 \theta = 1$, to transform it into an equation involving only one function of θ. Replace $\sin^2$ by $1 - \cos^2 \theta$.

$$\sin^2 \theta + 2 \cos \theta - 1 = 0$$

$$\cancel{1} - \cos^2 \theta + 2 \cos \theta - \cancel{1} = 0.$$

Factor out $\cos \theta$

$$\cos \theta \, (2 - \cos \theta) = 0.$$

Whenever a product of two numbers $ab = 0$, either $a = 0$ or $b = 0$, hence $\cos \theta = 0$ or $2 - \cos \theta = 0$. Thus $\cos \theta = 0$ or $\cos \theta = 2$.

Now there are two angles in the range $0 \le \theta < 2\pi$ for which $\cos \theta = 0$ namely

$$\theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}.$$

But, since a cosine of an angle can never exceed unity, the relation $\cos \theta = 2$ does not yield a value of θ. Hence we have just two solutions, as given above. It is easy to check these solutions.

Check: $\sin^2 \theta + 2 \cos \theta - 1 = 0$

$$(\sin \theta)^2 + 2 \cos \theta - 1 = 0$$

For $\theta = \frac{\pi}{2}$

$$(\sin \pi/2)^2 + 2 \cos \frac{\pi}{2} - 1 = 0$$

$$1 + 2 \cdot 0 - 1 = 0$$
$$0 = 0 \checkmark$$

For $\theta = \frac{3}{2}\pi$

$$(\sin 3/2\pi)^2 + 2 \cos \frac{3}{2}\pi - 1 = 0$$

$$(-1)^2 + 2 \cdot 0 - 1 = 0$$
$$0 = 0 \checkmark$$

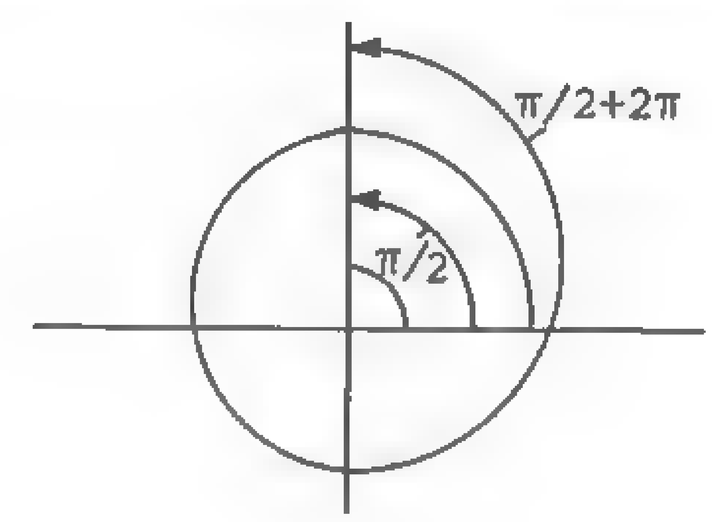
• **PROBLEM 4-13**

Find the solution set of $2 \cos^2 x - 5 \cos x + 2 = 0$.

Solution: Factoring, we obtain $(\cos x - 2)(2 \cos x - 1) = 0$. Setting each factor equal to zero, we obtain $\cos x = 2$ and $\cos x = 1/2$. There is no value of x satisfying the first factor because the range of values for $\cos x$ is from -1 to $+1$; that is, $-1 \le \cos x \le 1$. Therefore, the solution set of the first factor is the empty set. For the second factor the solution set is $\{x \mid x = \pi/3 + 2n\pi$ or $5\pi/3 + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots\}$. Since the first solution set is the empty set, the second set is the complete solution set.

• **PROBLEM 4-14**

Solve the equation $\sin^2 x - 4 \sin x + 3 = 0$.



Solution: Factoring the left side of the given equation into a product of two trigonometric functions

$$(\sin x - 3)(\sin x - 1) = 0 \qquad (1)$$

Whenever a product $ab = 0$ (where a and b are any two numbers) either $a = 0$ or $b = 0$. Hence, either

$$\sin x - 3 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
$$\sin x = 3 \quad \text{or} \quad \sin x = 1 ,$$

and so our desired solution set is the union

$$\{\sin x = 3\} \cup \{\sin x = 1\}. \qquad (2)$$

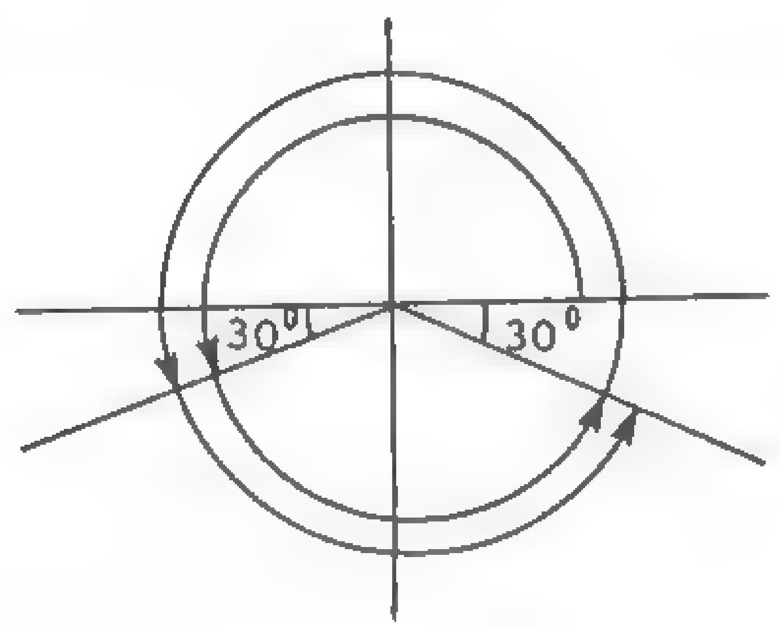
The first of these sets is the empty set because $\sin x$ takes on values only between -1 and 1; that is, $-1 \le \sin x \le 1$. For the second set, $\sin \pi/2 = 1$. Note that any integral multiple of 2π added to $\pi/2$ will result in the same reference angle. Hence,

$$\sin \frac{\pi}{2} = \sin\left(\frac{\pi}{2} + 2\pi\right) = \sin\left(\frac{\pi}{2} - 2\pi\right) = \sin\left(\frac{\pi}{2} + 4\pi\right) = 1 .$$

Therefore, the solution set for the second set is $\left\{\frac{\pi}{2} + 2\pi k\right.$, where k is $\left. 0, \pm 1, \pm 2, \ldots\right\}$. Since the first set has no solution set, the solution set for the second set is the solution set of the given equation. Hence, $\left\{\frac{\pi}{2} + 2\ k\right.$, where $\left. k = 0, \pm 1, \pm 2\right\}$ is the solution set of the given equation.

• PROBLEM 4-15

> Determine all angles x, $0^\circ \le x < 360^\circ$, such that $\sin 2x = -\frac{1}{2}$.



Solution: To determine all values of x such that $0^\circ \le x < 360^\circ$ and $\sin 2x = -\frac{1}{2}$, we must determine all values of $2x$ such that

$$2 \cdot 0^\circ \le 2 \cdot x < 2 \cdot 360^\circ \quad \text{and } \sin 2x = -\tfrac{1}{2},$$

or all values of $2x$ must be determined such that

$$0^\circ \le 2x < 720^\circ \quad \text{and} \quad \sin 2x = -\tfrac{1}{2}.$$

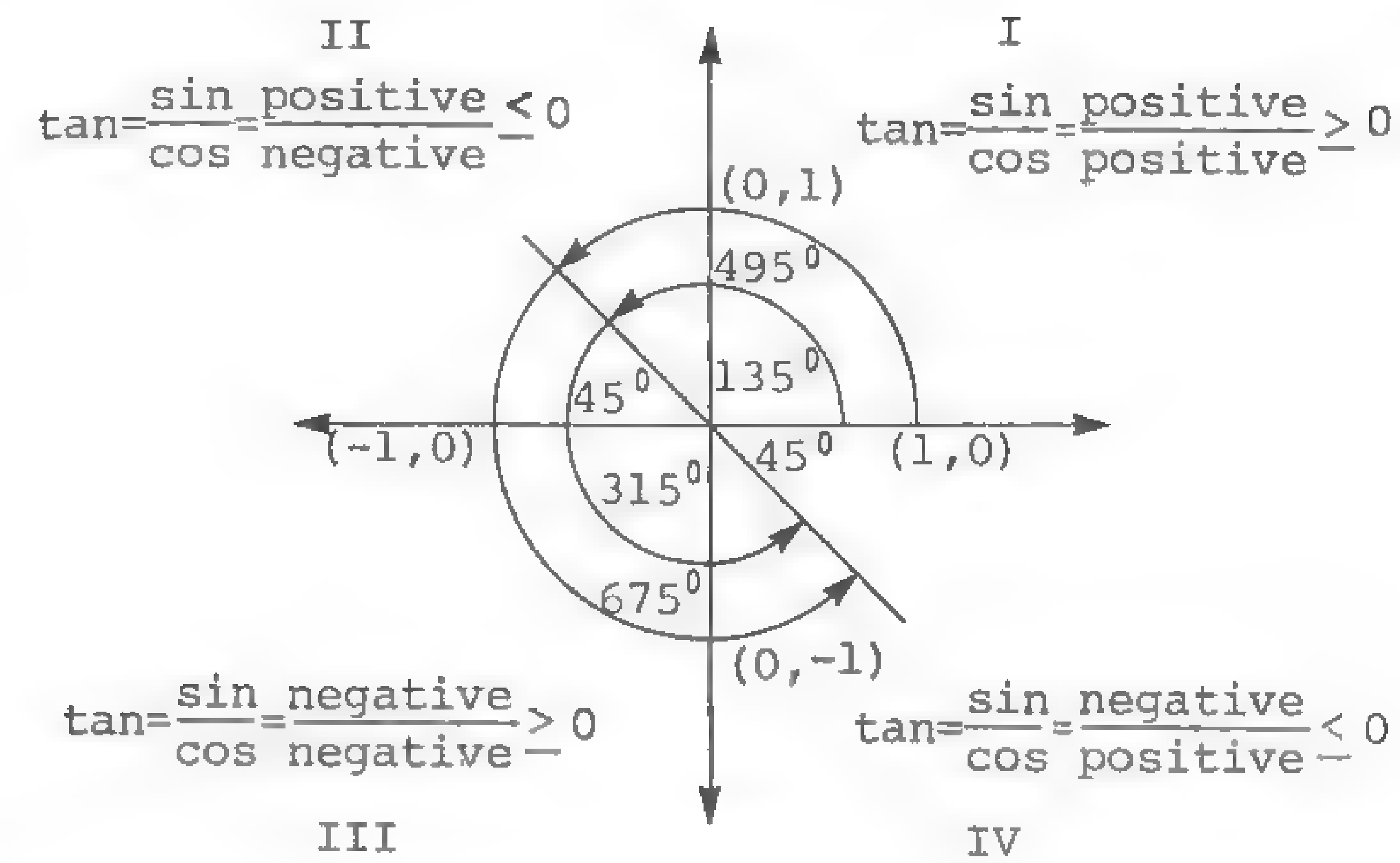
Since the sine function is negative in only the third and fourth quadrants, the angle $2x$ may lie in only these two quadrants. Also, since the $\sin 30^\circ = -\frac{1}{2}$ in the third and fourth quadrants, any angle with a reference angle of 30° will satisfy the equation $\sin 2x = -\frac{1}{2}$. The angles that satisfy this equation and which are in the range of $0^\circ \le 2x < 720^\circ$ are $180^\circ + 30^\circ = 210^\circ$, $360^\circ - 30^\circ = 330^\circ$, $360^\circ + 210^\circ = 570^\circ$, and $720^\circ - 30^\circ = 690^\circ$ (see diagram). Therefore, $2x = 210^\circ$ or $x = 105^\circ$, $2x = 330^\circ$ or $x = 165^\circ$, $2x = 570^\circ$ or $x = 285^\circ$, and $2x = 690^\circ$ or $x = 345^\circ$. Hence, the solutions of the equation are $x = 105^\circ$, $x = 165^\circ$, $x = 285^\circ$, and $x = 345^\circ$. These solutions are checked by substituting each of them into the equation.

$$\sin 2(105^\circ) = \sin 210^\circ = \sin 30^\circ = -\tfrac{1}{2}\checkmark$$
$$\sin 2(165^\circ) = \sin 330^\circ = \sin 30^\circ = -\tfrac{1}{2}\checkmark$$
$$\sin 2(285^\circ) = \sin 570^\circ = \sin 30^\circ = -\tfrac{1}{2}\checkmark$$
$$\sin 2(345^\circ) = \sin 690^\circ = \sin 30^\circ = -\tfrac{1}{2}\checkmark .$$

Note that the angles 210°, 333°, 570°, and 690° lie in either the third or fourth quadrant, in which the sine function is always negative.

• PROBLEM 4-16

> Determine all values of x such that $0^\circ \le x < 360^\circ$ and $\tan 2x = -1$.



Solution: To determine all angles x such that $0^\circ \le x < 360^\circ$, we must find all angles 2x such that $2 \cdot 0^\circ \le 2 \cdot x < 2 \cdot 360^\circ$ or $0^\circ \le 2x < 720^\circ$. Now determine in what quadrants tan is negative, to conform to the equation $\tan \theta = -1$. (See Figure). Thus, the desired angles lie in the second and fourth quadrants. For $\tan \theta = \sin \theta / \cos \theta = 1$ to be true, either $\sin \theta$ and $\cos \theta$ both equal 1, or $\sin \theta = \cos \theta$. Since sine and cosine are never equal to 1 simultaneously, we seek the angle for which sin and cos are equal. 45° satisfies this relation:

$$\frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

We must pick angles of 45° in the 2nd and 4th quadrants. They are 135° in the second quadrant, and 315° in the fourth quadrant. This is through one revolution or 360°. Through a second revolution (720°) the chosen angles are 495° in the second quadrant and 675° in the fourth (see Figure).

The angles are 135°, 315°, 495°, 675°. According to the equation each angle is twice the required angle. Therefore, the values of x which we are interested in are

$$\frac{135^\circ}{2} = 67.5^\circ$$

$$\frac{315^\circ}{2} = 157.5^\circ$$

$$\frac{495^\circ}{2} = 247.5^\circ$$

$$\frac{675^\circ}{2} = 337.5^\circ$$

• PROBLEM 4-17

Find the solution set of $5\tan^2\alpha - 2\tan\alpha - 1 = 0$.

Solution: Use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to find $\tan\alpha$. For the given equation, a = 5, b = -2, c = -1, and we solve for $x = \tan\alpha$.

$$\tan\alpha = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(-1)}}{2(5)} = \frac{2 \pm \sqrt{4 + 20}}{10} = \frac{2 \pm \sqrt{24}}{10}$$

$$= \frac{2 \pm \sqrt{4}\sqrt{6}}{10}$$

$$\tan\alpha = \frac{2 \pm 2\sqrt{6}}{10} = \frac{2 \pm 2(2.449)}{10}$$

$$\tan\alpha = \frac{2 \pm 4.8989}{10}$$

Therefore, $\tan\alpha = \dfrac{2 + 4.899}{10}$ and $\tan\alpha = \dfrac{2 - 4.899}{10}$

$= \dfrac{6.899}{10}$ $\qquad$ $= \dfrac{-2.899}{10}$

$\tan\alpha = 0.6899 \quad (1) \qquad \tan\alpha = -0.2899 \quad (2)$

For the $\tan\alpha = 0.6899$, α lies in the first and third quadrants since $\tan\alpha > 0$ in these quadrants. For $\tan\alpha = 0.6899$, $\alpha = 34^\circ\ 36'$ (an angle in the first quadrant) and $\alpha = 214^\circ\ 36'$ (an angle in the third quadrant). Note that the reference angle of $214^\circ\ 36'$ is $34^\circ\ 36'$, $214^\circ\ 36' - 180^\circ = 34^\circ\ 36'$. Hence, the solution set given by equation (1) is

$$\{\alpha | \alpha = 34^\circ 36' + n\pi,\ n = 0, \pm 1, \pm 2, \ldots\}.$$

For the $\tan\alpha = -0.2899$, α lies in the second and fourth quadrants since $\tan\alpha < 0$ in these quadrants. For $\tan\alpha = -0.2899$, $\alpha = 163^\circ 50'$. Note that the angle $163^\circ 50'$ is a reference angle for $16^\circ 10'$, $180^\circ - 163^\circ 50' = 16^\circ 10'$. Hence, the solution set given by equation (2) is

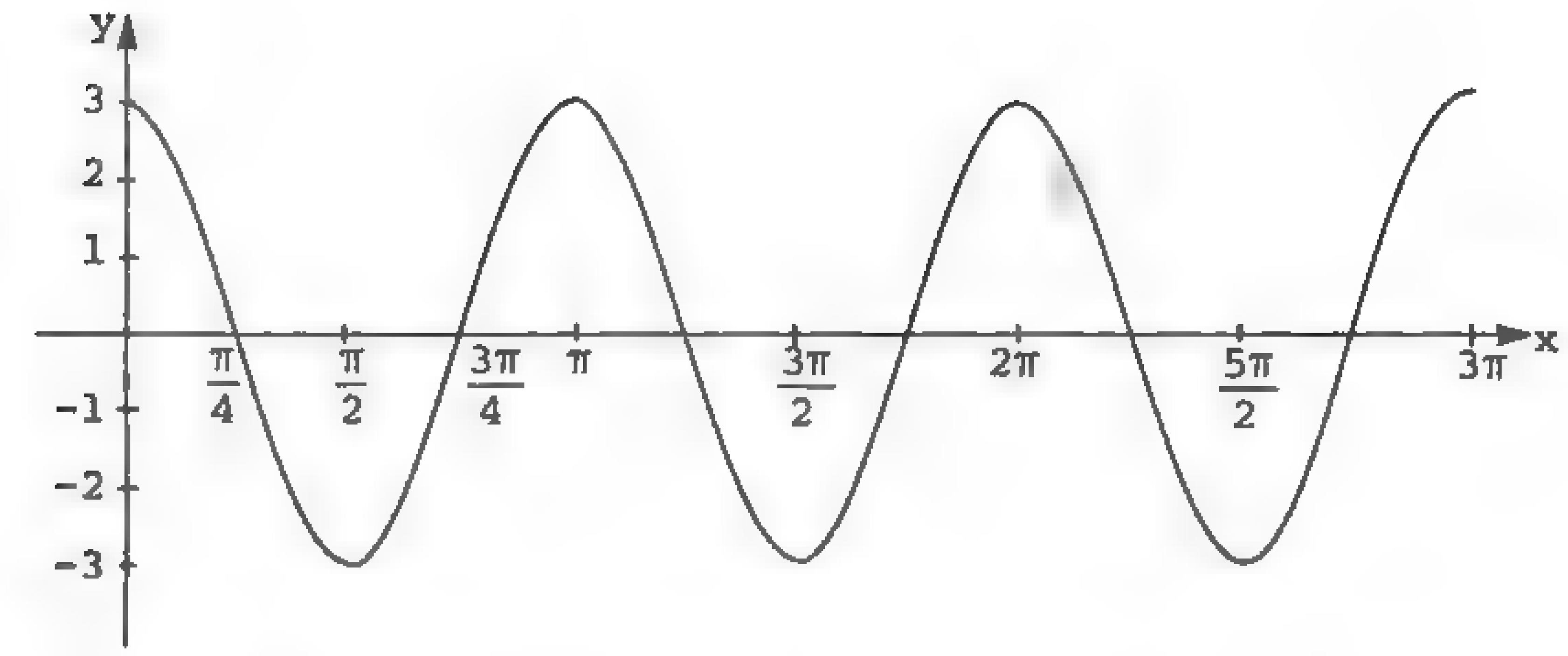
$$\{\alpha | \alpha = 163^\circ 50' + n\pi,\ n = 0, \pm 1, \pm 2, \ldots\}.$$

Therefore, the entire solution set is

$$\{\alpha | \alpha = 34^\circ 36' + n\pi \text{ or } \alpha = 163^\circ 50' + n\pi,\ n = 0, \pm 1, \pm 2, \ldots\}.$$

• PROBLEM 4-18

Sketch three periods of the graph y = 3 cos 2x.



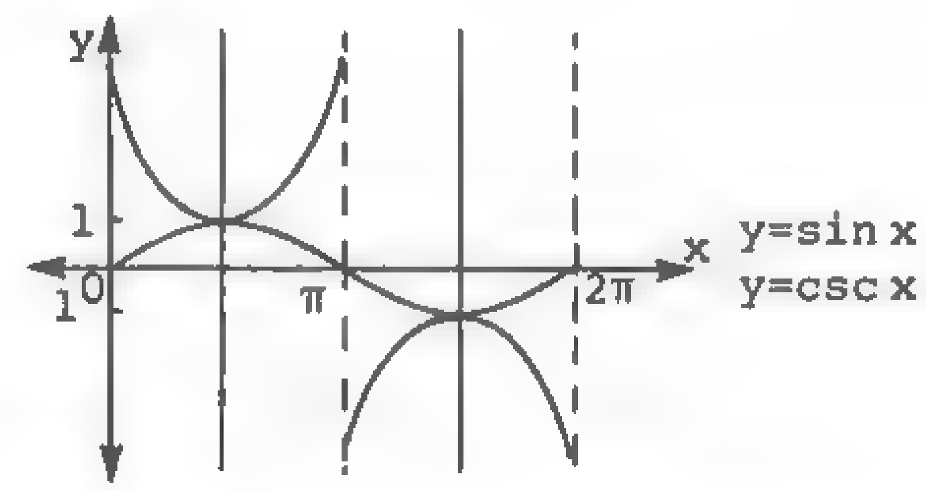
Solution: The coefficient of the function is 3, which means that the maximum and minimum values are 3 and - 3, respectively. The period of the cosine function is the coefficient of x multiplied by $\frac{\pi}{2}$ radians. Therefore, the period of the cosine function given in this problem is

$$2\left(\frac{\pi}{2}\text{ radians}\right) = \pi \text{ radians}$$

and with this knowledge, we sketch the curve as in the Figure.

• PROBLEM 4-19

Graph $y = \csc x,\ 0 \le x \le 2\pi$.



Solution: To plot points for the function cosecant of x, first find the y-values of the reciprocal function, the sine of the angle x.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin x	0	1	0	-1	0
csc x	not defined	1	not defined	-1	not defined

Since the sine and cosecant are reciprocals, we state the following conclusions based on properties of real numbers.

(1) For $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$ and $\csc x \ge 1$.

In fact, as sin x increases, csc x decreases. For example:

$\sin 0^\circ = \sin 0 = 0$ $\qquad\qquad$ $\csc 0 = \text{undefined}$

$$\sin 30^{\circ} = \sin\frac{\pi}{6} = \frac{1}{2} = .5000 \qquad \csc\frac{\pi}{6} = 2$$

$$\sin 45^{\circ} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = .7070 \qquad \csc\frac{\pi}{4} = \sqrt{2} \approx 1.414$$

$$\sin 90^{\circ} = \sin\frac{\pi}{2} = 1.000 \qquad \csc\frac{\pi}{2} = 1$$

(2) For $\frac{\pi}{2} \le x \le \pi$, sin x decreases from 1 to 0. Hence, csc x will increase from 1 to very large values. We can observe this from specific examples:

$$\sin\frac{\pi}{2} = 1 \qquad \csc\frac{\pi}{2} = 1$$

$$\sin\frac{2}{3}\pi = \frac{\sqrt{3}}{2} \approx .87 \qquad \csc\frac{2}{3}\pi \approx 1.15$$

$$\sin\frac{3}{4}\pi \approx .707 \qquad \csc\frac{3}{4}\pi \approx 1.4$$

$$\sin\frac{5}{6}\pi = .500 \qquad \csc\frac{5}{6}\pi = 2$$

$$\sin \pi = 0 \qquad \csc \pi = \text{undefined}$$

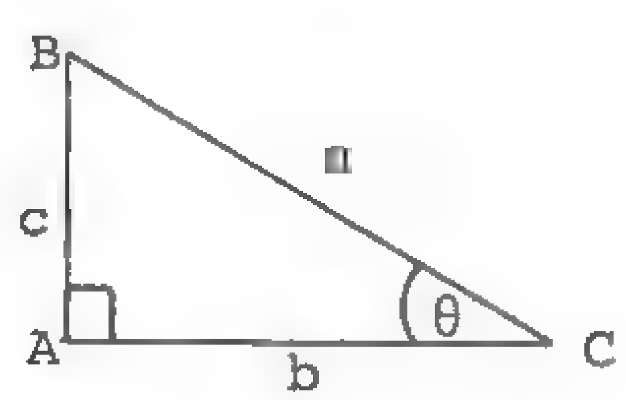
(3) For $\pi \le x \le \frac{3\pi}{2}$, sin x decreases from 0 to -1. Hence, csc x will be increasing and will increase from very large negative values to -1.

(4) For $\frac{3\pi}{2} \le x \le 2\pi$, $-1 \le \sin x \le 0$, and the graph will be increasing. Hence, csc x will decrease from -1 to very large negative values. The student can verify conclusions (3) and (4) in a similar manner to that used for (1) and (2), that is, choosing specific angles between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and then between $\frac{3\pi}{2}$ and 2π.

The graphs of both functions are shown in the accompanying figure. Note that the range of the sine function is $-1 \le \sin x \le 1$, but the range of the cosecant function is $\csc x \ge 1$ or $\csc x \le -1$.

• PROBLEM 4-20

> Show that $\frac{\cos\theta}{\sin\theta} = \cot\theta$ is an identity.



Solution: In the diagram $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{a}$, and

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{c}{a}.$$

Thus $\frac{\cos\theta}{\sin\theta} = \frac{\frac{b}{a}}{\frac{c}{a}}$. Multiplying this fraction by $\frac{a}{a}$ (= 1) gives us

$$\frac{\frac{b}{a}}{\frac{c}{a}} \cdot \frac{a}{a} = \frac{b}{c}.$$

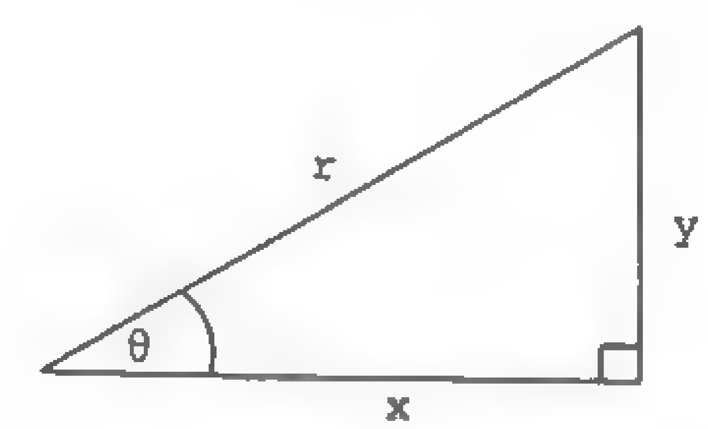
We observe that b is the adjacent side of angle θ and c is the opposite side. Hence, $\frac{b}{c} = \frac{\text{adjacent}}{\text{opposite}} = \cot\theta$, by definition.

• PROBLEM 4-21

> Show that $\sec^2\theta - \tan^2\theta = 1$ is an identity.

Solution: Given a right triangle (see the accompanying figure) we have:

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}$$



$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{r}{x}$$

Substitute these expressions into the identity:

$$\sec^2\theta - \tan^2\theta = \left(\frac{r}{x}\right)^2 - \left(\frac{y}{x}\right)^2 = \frac{r^2}{x^2} - \frac{y^2}{x^2} = \frac{r^2 - y^2}{x^2}$$

By the Pythagorean Theorem, $r^2 = x^2 + y^2$; substitute x^2 for $r^2 - y^2$; since $x^2 + y^2 = r^2$,

$$x^2 = r^2 - y^2.$$

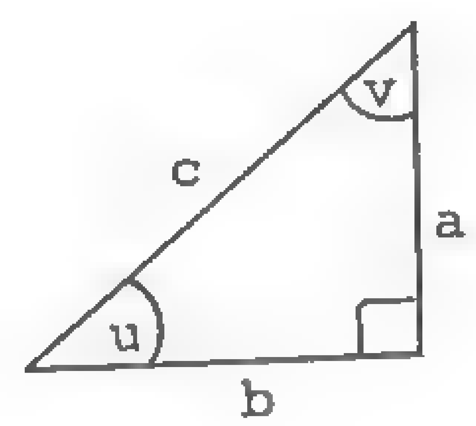
Thus, $\sec^2\theta - \tan^2\theta$

$$= \frac{x^2}{x^2} = 1$$

• PROBLEM 4-22

If u and v are two numbers such that $u + v = \frac{1}{2}\pi$,

show that $\sin^2 u + \sin^2 v = 1$.



Solution: We know the following trigonometric identity,

$\sin^2 u + \cos^2 u = 1$.

Now, if $u + v = \frac{\pi}{2}$, these angles are complementary, that is, the sum of these angles is $90° = \frac{\pi}{2}$. From the accompanying figure, it is seen that

$$\sin v = \frac{b}{c} = \cos u$$

Substitute this relation into the identity to obtain:

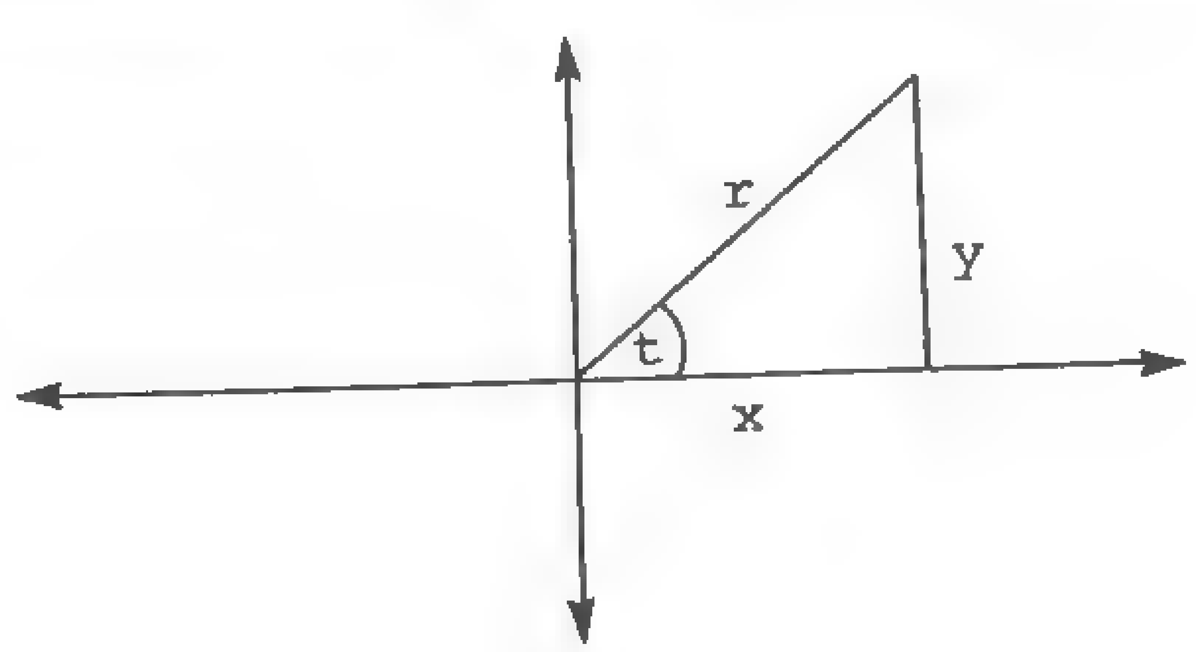
$$\sin^2 u + \cos^2 u = (\sin u)^2 + (\cos u)^2$$

$$= (\sin u)^2 + (\sin v)^2$$

$$= \sin^2 u + \sin^2 v = 1.$$

• PROBLEM 4-23

Show that $\tan^2 t + 1 = \sec^2 t$.



Solution: This equation is meaningless if the X-coordinate of $P(t)$ is 0, since, from the accompanying figure,

$$\tan t = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x} = \text{undefined when } x = 0$$

and

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{\text{adjacent side}}{\text{hypotenuse}}} = \frac{1}{\frac{x}{r}}$$

$$= \frac{r}{x} = \text{undefined when } x = 0$$

Furthermore, if the $\cos t$ is zero, then the $\sec t$ is undefined. We assume that it is not. In other words, $\cos t \neq 0$ for each number in which we are interested. Thus we may divide both sides of the elementary identity $\sin^2 t + \cos^2 t = 1$ by $\cos^2 t$ to obtain the equation:

$$\frac{\sin^2 t}{\cos^2 t} + 1 = \frac{1}{\cos^2 t}$$

Substitute $\tan t = \frac{\sin t}{\cos t}$ and $\sec t = \frac{1}{\cos t}$, to obtain:

$$\tan^2 t + 1 = \sec^2 t.$$

• PROBLEM 4-24

Prove the identity $\csc 2x = \frac{\csc x}{2\cos x}$.

Solution: Starting with the right side of the identity,

$$\frac{\csc x}{2\cos x} = \frac{1/\sin x}{2\cos x}, \text{ since } \csc x = \frac{1}{\sin x}. \quad \text{Hence,}$$

$$\frac{\csc x}{2\cos x} = \frac{1}{\sin x} \cdot \frac{1}{2\cos x} = \frac{1}{2\sin x \cos x}.$$

Using the double-angle formula, $\sin 2\alpha = 2\sin\alpha\cos\alpha$,

$$\frac{\csc x}{2\cos x} = \frac{1}{2\sin x\cos x} = \frac{1}{\sin 2x}.$$

Again, since $\csc x = \frac{1}{\sin x}$, $\frac{\csc x}{2\cos x} = \frac{1}{\sin 2x} = \csc 2x$. Since we have proved the left side equal to the right,

$$\frac{\csc x}{2\cos x} = \csc 2x.$$

• PROBLEM 4-25

Prove the identity $\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \sec^2\theta$.

Solution: The fraction $\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b}$, by the definition of addition of fractions. Thus,

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}$$

$$= \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1,$$

since the fraction $\frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2$ by definition, and $\frac{\cos^2\theta}{\cos^2\theta} = 1$.

$\frac{\sin\theta}{\cos\theta} = \tan\theta$ by definition. Thus we obtain:

$$\tan^2\theta + 1;$$

and using the trignometric identity $\tan^2\theta + 1 = \sec^2\theta$, we have:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \sec^2\theta,$$

our desired result.

• PROBLEM 4-26

Prove the identity $\frac{1 - \sin^2\alpha}{\sin^2\alpha} = \cot^2\alpha$.

Solution: The left member is the more complicated of the two sides of the identity. Operating only on the left member, we obtain

$$\frac{1 - \sin^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha} - \frac{\sin^2\alpha}{\sin^2\alpha}$$

$$= \frac{1}{\sin^2\alpha} - 1.$$

Since $\csc\theta = \frac{1}{\sin\theta}$ or $\csc^2\theta = \frac{1}{\sin^2\theta}$ for any angle θ, then

$$\frac{1 - \sin^2\alpha}{\sin^2\alpha} = \csc^2\alpha - 1.$$

Also, since $\csc^2\theta - \cot^2\theta = 1$ or $\csc^2\theta - 1 = \cot^2\theta$ for any angle θ, then

$$\frac{1 - \sin^2\alpha}{\sin^2\alpha} = \cot^2\alpha.$$

• PROBLEM 4-27

Prove that

$$\frac{\cos^3 x - \cos x + \sin x}{\cos x} = \tan x - \sin^2 x$$

is an identity.

Solution: If we put each term of the numerator separately over the denominator, we have

$$\frac{\cos^3 x - \cos x + \sin x}{\cos x} = \frac{\cos^3 x}{\cos x} - \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}.$$

But $\frac{\cos^3 x}{\cos^1 x} = \cos^{3-1} x = \cos^2 x$

$\frac{\cos x}{\cos x} = 1$

and $\frac{\sin x}{\cos x} = \tan x$.

Thus, replacing these values we obtain:

$$= \cos^2 x - 1 + \tan x.$$

Recall the identity $\sin^2\theta + \cos^2\theta = 1$.

Subtracting $\sin^2\theta$ from both sides gives us

$$\cos^2\theta = 1 - \sin^2\theta,$$

and subtracting 1 from both sides we obtain

$$\cos^2\theta - 1 = -\sin^2\theta;$$

thus replacing $\cos^2 x - 1$ by $-\sin^2 x$ we have:

$$\frac{\cos^3 x - \cos x + \sin x}{\cos x} = -\sin^2 x + \tan x$$

$$= \tan x - \sin^2 x.$$

• PROBLEM 4-28

Show that $\tan t + \cot t = \csc t \sec t$.

Solution: Since $\tan t = \frac{\sin t}{\cos t}$ and $\cot t = \frac{\cos t}{\sin t}$, by substitution we have:

$$\tan t + \cot t = \frac{\sin t}{\cos t} + \frac{\cos t}{\sin t}$$

Since multiplying by $1 \equiv \frac{\sin t}{\sin t}$ and $1 \equiv \frac{\cos t}{\cos t}$ does not alter the value of either fraction, we perform this multiplication, and obtain:

$$\frac{\sin t}{\cos t}\left(\frac{\sin t}{\sin t}\right) + \frac{\cos t}{\sin t}\left(\frac{\cos t}{\cos t}\right)$$

$$= \frac{\sin^2 t}{\sin t \cos t} + \frac{\cos^2 t}{\sin t \cos t}$$

$$= \frac{\sin^2 t + \cos^2 t}{(\sin t)(\cos t)}$$

Now, using the trigonometric identity, $\sin^2 t + \cos^2 t = 1$, we obtain:

$$\frac{1}{(\sin t)(\cos t)};$$

and, since $\frac{1}{\sin t} = \csc t$ and $\frac{1}{\cos t} = \sec t$, we obtain the desired result:

$$\tan t + \cot t = \csc t \sec t.$$

• PROBLEM 4-29

Prove that

$$\frac{1}{\sec A - \tan A} = \sec A + \tan A$$

is an identity.

Solution: If we multiply numerator and denominator of the left member of the given equation by $\sec A + \tan A$, we have

$$\frac{1}{\sec A - \tan A} = \left(\frac{1}{\sec A - \tan A}\right)\left(\frac{\sec A + \tan A}{\sec A + \tan A}\right)$$

(Note that $(\sec A + \tan A)/(\sec A + \tan A) = 1$, and therefore does not alter the equation)

$$= \frac{\sec A + \tan A}{(\sec A - \tan A)(\sec A + \tan A)}$$

$$= \frac{\sec A + \tan A}{\sec^2 A - \tan^2 A}$$

Recall the trigonometric identity $1 + \tan^2 \theta = \sec^2 \theta$.

Subtracting $\tan^2 \theta$ from both sides we obtain $1 = \sec^2 \theta - \tan^2 \theta$. Thus replacing $\sec^2 A - \tan^2 A$ by 1, we obtain:

$$\frac{1}{\sec A - \tan A} = \frac{\sec A + \tan A}{1} = \sec A + \tan A.$$

• PROBLEM 4-30

Prove the identity $\frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$.

Solution: One side of this identity is as complicated as the other so that it makes no difference which side is used. The illustration uses both sides.

(a) $$\frac{1 - \cos\theta}{\sin\theta} = \frac{(1 - \cos\theta)}{\sin\theta}\,\frac{(1 + \cos\theta)}{(1 + \cos\theta)}$$

$$= \frac{1 - \cos\theta + \cos\theta - \cos^2\theta}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{1 - \cos^2\theta}{\sin\theta(1 + \cos\theta)}$$

Since $\cos^2\theta + \sin^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$, then

$$\frac{1 - \cos\theta}{\sin\theta} = \frac{1 - \cos^2\theta}{\sin\theta(1 + \cos\theta)} = \frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{\sin\theta}{1 + \cos\theta}$$

Note that this method starts with the left side of the identity to be proved. The following is another method which can be used to prove the identity.

(b) $$\frac{\sin\theta}{1 + \cos\theta} = \frac{\sin\theta(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}$$

$$= \frac{\sin\theta(1 - \cos\theta)}{1 + \cos\theta - \cos\theta - \cos^2\theta}$$

$$= \frac{\sin\theta(1 - \cos\theta)}{1 - \cos^2\theta}$$

Again, since $\sin^2\theta = 1 - \cos^2\theta$,

$$\frac{\sin\theta}{1 + \cos\theta} = \frac{\sin\theta(1 - \cos\theta)}{1 - \cos^2\theta}$$

$$= \frac{\sin\theta(1 - \cos\theta)}{\sin^2\theta}$$

$$= \frac{1 - \cos\theta}{\sin\theta}$$

Note that this second method starts with the right side of the identity to be proved.

• PROBLEM 4-31

Prove that

$$\frac{\cos A}{\csc A - 1} + \frac{\cos A}{\csc A + 1} = 2 \tan A$$

is an identity.

Solution: The left member is the more complicated; hence, we shall work with it and begin by performing the indicated addition. The lowest common denominator is $(\csc A - 1)(\csc A + 1) = \csc^2 A - 1$. Thus

$$\frac{\cos A}{\csc A - 1} + \frac{\cos A}{\csc A + 1} = \left(\frac{\csc A + 1}{\csc A + 1}\right)\frac{\cos A}{\csc A - 1} + \left(\frac{\csc A - 1}{\csc A - 1}\right)\frac{\cos A}{\csc A + 1}$$

$$= \frac{(\csc A + 1)\cos A}{\csc^2 A - 1} + \frac{(\csc A - 1)\cos A}{\csc^2 A - 1}$$

$$= \frac{\csc A \cos A + \cos A}{\csc^2 A - 1} + \frac{\csc A \cos A - \cos A}{\csc^2 A - 1}$$

$$= \frac{\csc A \cos A + \cos A + \csc A \cos A - \cos A}{\csc^2 A - 1}$$

$$= \frac{\csc A \cos A + \csc A \cos A}{\csc^2 A - 1}$$

Recall the trignometric identity $\csc^2 A - 1 = \cot^2 A$,

$$= \frac{2 \cos A \csc A}{\cot^2 A}$$

replace $\csc A$ by $\frac{1}{\sin A}$,

$$= \frac{(2 \cos A)/(\sin A)}{\cot^2 A}$$

replace $\frac{\cos A}{\sin A}$ by $\cot A$,

$$= \frac{2 \cot A}{\cot^2 A}$$

cancelling out $\cot A$,

$$= \frac{2}{\cot A}$$

Replace $\cot A$ by $\frac{1}{\tan A}$, $= \frac{2}{\frac{1}{\tan A}}$

Multiply numerator and denominator by $\tan A$,

$$= 2 \tan A$$

We have thus proved

$$\frac{\cos A}{\csc A - 1} + \frac{\cos A}{\csc A + 1} = 2 \tan A$$ is an identity.

• PROBLEM 4-32

Prove that the following equation is an identity:

$$\frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x}$$

Solution: This problem may be approached in a variety of ways. One method is based on the fact that two fractions are equal if their cross products are equal. That is,

$$\frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x} \quad \text{if } (1 - \sin x)(1 + \sin x) = (\cos x)(\cos x).$$

$$(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$$

Now, recall the trigonometric identity $\cos^2 \theta = 1 - \sin^2\theta$; thus $(1 - \sin x)(1 + \sin x) = \cos^2 x = (\cos x)\cdot(\cos x)$, and since the cross products are equal, we have proven the original fractions equivalent.

• PROBLEM 4-33

Prove the identity: $\sec A \csc A = \tan A + \cot A$.

Solution: One approach to the proof of identities, when many functions are involved, is to express the given functions in terms of fewer functions. In this case, suppose we express each of the given trigonometric functions in terms of sine and cosine functions. We will work in parallel columns, with each side of the given equation:

Since $\sec A = 1/\cos A$ and $\csc A = 1/\sin A$,

$\sec A \csc A$

$$= \frac{1}{\cos A} \cdot \frac{1}{\sin A}$$

$$= \frac{1}{\cos A \sin A}$$

Since $\tan A = \sin A/\cos A$ and $\cot A = 1/\tan A = 1/\sin A/\cos A = \cos A/\sin A$, $\tan A + \cot A$

$$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}.$$

To combine these fractions, we convert them into fractions with the least common denominator (LCD) $\cos A \sin A$; thus:

$$= \left(\frac{\sin A}{\sin A} \ \frac{\sin A}{\cos A}\right) + \left(\frac{\cos A}{\cos A} \ \frac{\cos A}{\sin A}\right)$$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}.$$

Recall the trigonometric identity $\sin^2 A + \cos^2 A = 1$; replacing $\sin^2 A + \cos^2 A$ by 1 we obtain:

$$= \frac{1}{\cos A \sin A}$$

Now, since we have proved that both sides of the given equation are equal to the same expression, we are tempted to say that they are therefore equal to each other, and that we have therefore proved what we set out to prove.

We have indeed, except for one detail. We have not considered the values of A for which the given expressions and those which we substituted are meaningful.

This aspect of the proof of a trigonometric identity rarely leads to trouble, and may therefore usually be omitted. The careful student, however, will want to be prepared to investigate this question.

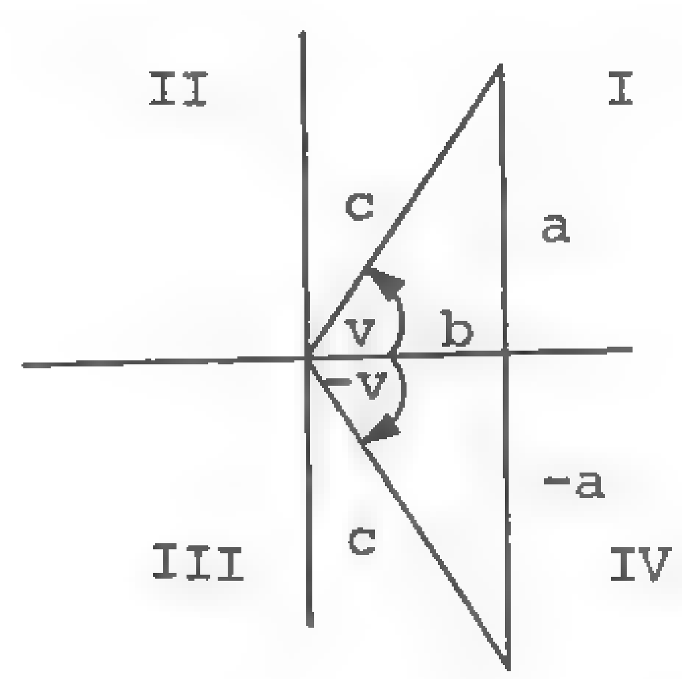
Thus we note that sec A and tan A are defined if and only if A is a real number of degrees not an odd multiple of 90; csc A and cot A are defined if and only if A is a real number of degrees not an even multiple of 90. Both sides of the equation are therefore defined if and only if A is a real number of degrees not an integer multiple of 90.

Each of the substitutions made in the parallel columns above is valid if A is such a number.

Therefore sec A csc A and tan A cot A are equal whenever both are defined, and the equation sec A csc A = tan A + cot A is an identity.

• PROBLEM 4-34

Show that $\tan(-v) = -\tan v$ for every number v in the domain of the tangent function.



Solution: We have

$$\tan(-v) = \frac{\sin(-v)}{\cos(-v)}$$

Now, let us examine $\sin(-v)$ and $\cos(-v)$. We know that in drawing the negative angles we rotate clockwise on the coordinate axes.

Thus, the value of the negative angle is the same as that of the reference angle in quadrant I, but it may have a different sign. Therefore, sin v and $\sin(-v)$ have the value $\frac{a}{c}$, but for angle $-v$, the opposite side is $-a$, thus $\sin(-v) = \frac{-a}{c} = -\sin v$. Cos v and $\cos(-v)$ have the value $\frac{b}{c}$, and $\cos v = \frac{b}{c}$, $\cos(-v) = \frac{b}{c}$; thus, $\cos(-v) = \cos v$. Therefore, $\frac{\sin(-v)}{\cos(-v)}$ can be written as:

$$\frac{-\sin v}{\cos v} = -\tan v.$$

• PROBLEM 4-35

Prove $\sin(45^\circ + x) + \sin(45^\circ - x) = \sqrt{2}\cos x$.

Solution: Use the formula for the sine of the sum and the difference of two angles.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Adding these two shows that $\cos A \sin B$ will cancel and the result becomes

$$2 \sin A \cos B \quad \text{with} \quad A = 45^\circ.$$

Therefore we have

$$\sin(45^\circ + x) + \sin(45^\circ - x) = 2 \sin 45^\circ \cos x$$
$$= \frac{2\sqrt{2}}{2} \cos x = \sqrt{2} \cos x.$$

• PROBLEM 4-36

Prove that $\frac{\cos\theta}{1 - \sin\theta} = \frac{1 + \sin\theta}{\cos\theta}$.

Solution: $\frac{\cos\theta}{1 - \sin\theta} = \frac{\cos\theta}{1 - \sin\theta} \cdot \frac{1 + \sin\theta}{1 + \sin\theta}$, since multiplication by

1 does not change the value of the fraction. Performing the multiplication we obtain: $\dfrac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$$= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}.$$

Since we are dividing by $1 - \sin^2\theta$, we must be sure $1 - \sin^2\theta \neq 0$, that is $1 \neq \sin^2\theta$. Thus $\sin\theta \neq 1, -1$.

We use the identity $1 - \sin^2\theta = \cos^2\theta$ to rewrite the fraction as follows: $\dfrac{\cos\theta(1+\sin\theta)}{\cos^2\theta} = \dfrac{\cos\theta(1+\sin\theta)}{(\cos\theta)(\cos\theta)}$, since

$$\cos^2\theta = (\cos\theta)(\cos\theta).$$

Finally, $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$. Since we are dividing by $\cos\theta$ we must exclude all $\cos\theta = 0$. Hence $\cos\theta \neq 0$. Thus, we have proven that $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$, with the restrictions

$$\cos\theta \neq 0,\ \sin\theta \neq 1, -1.$$

CHAPTER 5

VECTORS

• PROBLEM 5-1

Which of the following vectors are equal to $\overrightarrow{MN}$ if $M = (2, 1)$ and $N = (3, -4)$?
(a) $\overrightarrow{AB}$, where $A = (1, -1)$ and $B = (2, 3)$
(b) $\overrightarrow{CD}$, where $C = (-4, 5)$ and $D = (-3, 10)$
(c) $\overrightarrow{EF}$, where $E = (3, -2)$ and $F = (4, -7)$.

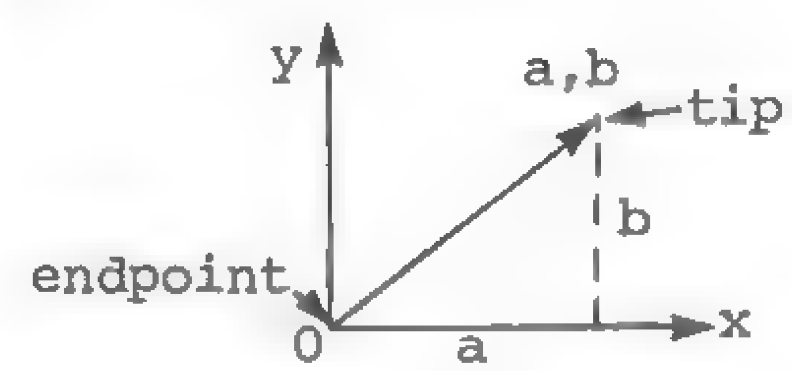


Fig. A: (a-0,b-0) represents the vector.

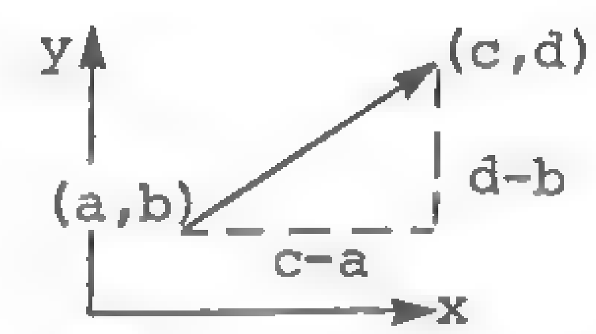


Fig. B: (c-a,d-b) represents the vector.

Solution: With each ordered pair in the plane there can be associated a vector from the origin to that point.

The vector is determined by subtracting the coordinates of the endpoint from the corresponding coordinates of the tip. As for $\overrightarrow{MN}$, the tip is the point corresponding to the second letter of the alphabetical notation, N, while the endpoint is the point corresponding to the first, M. In this problem the vectors are of a general nature wherein their endpoints do not lie at the origin.

We first find the ordered pair which represents $\overrightarrow{MN}$.

$$\overrightarrow{MN} = (3 - 2, -4 - 1) = (1, -5)$$

Now, we find the ordered pair representing each vector.

(a) $\overrightarrow{AB} = (2 - 1,\ 3 - (-1)) = (1,\ 4)$

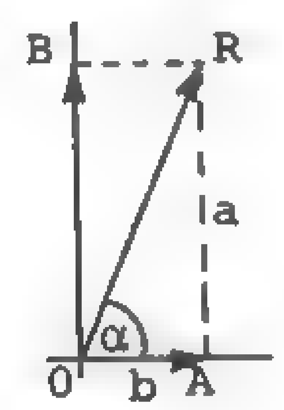
(b) $\overrightarrow{CD} = ((-3) - (-4),\ 10 - 5) = (1,\ 5)$

(c) $\overrightarrow{EF} = (4 - 3,\ -7 - (-2)) = (1,\ -5)$

Only $\overrightarrow{EF}$ and $\overrightarrow{MN}$ are equal.

• PROBLEM 5-2

A force of 315 lbs. is acting at an angle of 67° with the horizontal. What are its horizontal and vertical components?



Solution: Construct the figure shown.

OR = vector force = c.

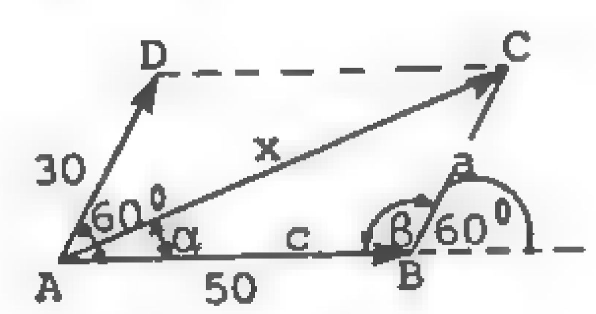
b = OA = horizontal component.

a = OB = vertical component.

In △OAR: c = 315; α = 67°.		
$\frac{a}{c} = \sin\alpha$, or $a = c\sin\alpha$.	$\log c = 2.49831$ $\log\sin\alpha = 9.96403 - 10$ $\log a = 2.46234$	$\boxed{a = 289.96\text{ lbs.}}$
$\frac{b}{c} = \cos\alpha$, or $b = c\cos\alpha$.	$\log c = 2.49831$ $\log\cos\alpha = 9.59188 - 10$ $\log b = 2.09019$	$\boxed{b = 123.08\text{ lbs.}}$

• PROBLEM 5-3

Two forces of 50 lbs. and 30 lbs. have an included angle of 60°. Find the magnitude and direction of their resultant.



Solution: Construct the parallelogram and label it as in the figure. Since $\overline{AD}$ is parallel to $\overline{BC}$ we have $\angle ABC = \beta = 180° - 60° = 120°$.

By the law of cosines:

$$x^2 = c^2 + a^2 - 2\,ac\cos\beta$$

$$= 2500 + 900 - 2(50)(30)(-\tfrac{1}{2})$$

$$= 2500 + 900 + 1500 = 4900.$$

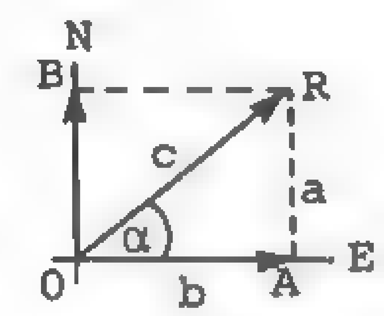
$$x = 70 \text{ lbs.}$$

$$\cos\alpha = \frac{x^2 + c^2 - a^2}{2xc} = \frac{4900 + 2500 - 900}{2(70)(50)} = \frac{13}{14} = .9286.$$

$$\alpha = 21°47'.$$

• PROBLEM 5-4

Two forces act simultaneously on a body free to move. One force of 112 lbs. is acting due east, while the other of 88 lbs. is acting due north. Find the magnitude and direction of their resultant.



Solution: Construct the figure shown.

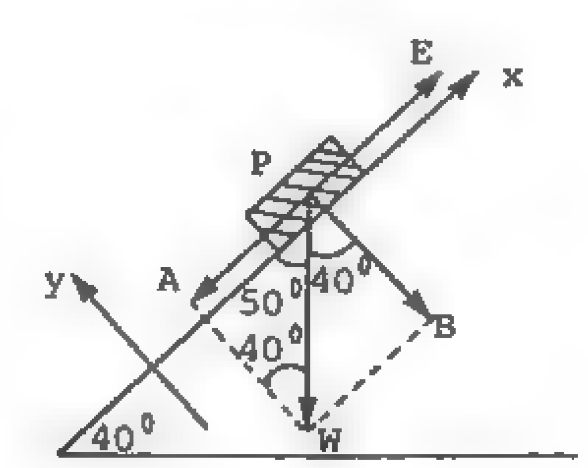
OA = b = 112 lbs.

OB = 88 lbs. = RA = a.

In △OAR: a = 88; b = 112.		
$\frac{a}{b} = \tan\alpha$.	$\log a = 1.94448$ $\log b = 2.04922$ $\log\tan\alpha = 9.89526 - 10$	$\boxed{\alpha = 38°9'25''}$
$\frac{a}{c} = \sin\alpha$, or $c = \frac{a}{\sin\alpha}$	$\log a = 11.94448 - 10$ $\log\sin\alpha = 9.79086 - 10$ $\log c = 2.15362$	$\boxed{c = 142.44}$
Therefore the resultant is 142.44 lbs. and its direction is 38°9′25″ north of east.		

• PROBLEM 5-5

Find the force required to prevent a 500-pound object from sliding down a 40° incline, disregarding friction.



Solution: The weight of the object acts as a 500-pound force vertically downward. The component of the force parallel to the plane tends to force the object down the incline. The component perpendicular to the plane tends to force the object against the plane. The required force is parallel to the plane, equal in magnitude and opposite in direction to the parallel component. Observing the figure, the triangle WPB is similar to the triangle made by the inclined plane and the ground. Angle WPB therefore equals 40°. It follows then that angle APW equals 50°, as it is complementary to angle WPB. By super posing the coordinate axes with the x axis parallel to the inclined plane, observe that the component of the force to be determined lies parallel to the inclined plane and can be found by multiplying the magnitude of weight by the cos of 50°. Here cos 50° can be calculated using the rule $\cos\ \theta$ = adjacent/hypotenuse for right triangles. The adjacent side is $\overrightarrow{AP}$ and the hypotenuse is $\overrightarrow{PW}$. Therefore $\cos 50^\circ = AP/PW$ or $(PW) \cos 50^\circ = AP$, where Ap and PW represent the magnitudes of the force vectors.

$$\overrightarrow{PW} = 500, \quad \sphericalangle APW = 90^\circ - 40^\circ = 50^\circ$$

$$\cos \sphericalangle APW = \frac{AP}{PW}, \quad \text{where } AP = |\overrightarrow{AP}|$$

$$AP = PW \cos \sphericalangle APW$$

$$AP = 500 \cos 50^\circ$$

$$AP = 500(0.6428) = 321$$

The required force $\overrightarrow{PE}$ is 321 pounds.

• PROBLEM 5-6

Find the magnitude and direction of the force necessary to counteract the effect of a force of 60 pounds and a force of 40 pounds that act on a point at an angle of 60° with each other.

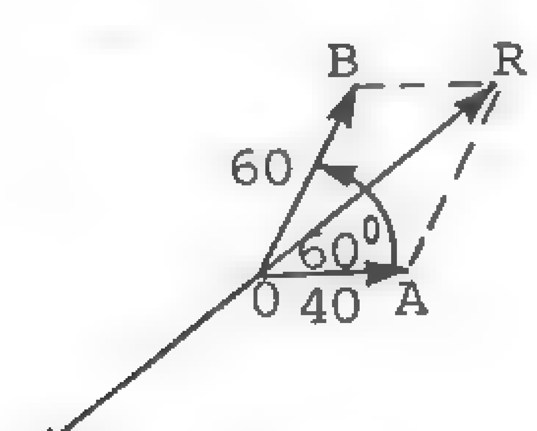


Fig. 1

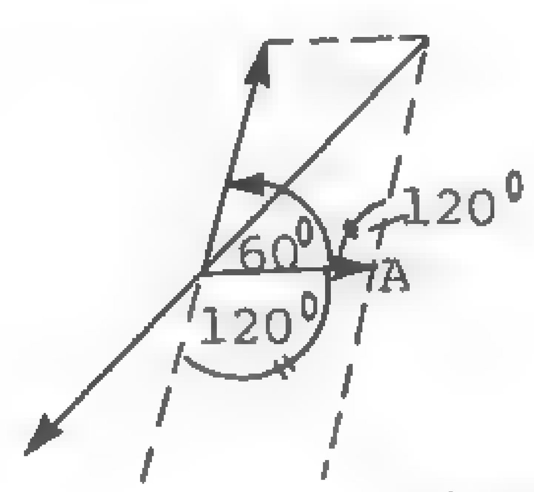


Fig. 2

Solution: The problem requires finding the equilibrant of the two forces. The equilibrant is equal in magnitude, but opposite in direction, to the resultant force. We find the resultant force of two vectors by using the parallelogram law of addition of vectors. Two vectors

are drawn and the parallelogram is completed. The resultant is drawn connecting the two opposite vertices (see figure 1). We indicate the magnitude of a vector as the corresponding segment. In the parallelogram, resultant $\overrightarrow{OR}$ may be found by solving triangle OAR.

$$OA = BR = 40$$

$$AR = OB = 60$$

$$\angle A = 180^\circ - 60^\circ = 120^\circ \text{(see figure 2)}$$

By the Law of Cosines:

$$(OR)^2 = (OA)^2 + (AR)^2 - [2(OA)(AR) \cos 120^\circ]$$

$$(OR)^2 = (40)^2 + (60)^2 - [(2)(40)(60)(-\tfrac{1}{2})]$$

$$(OR)^2 = 7{,}600$$

$$OR = 87$$

From the Law of Sines,

$$\sin\angle AOR = \frac{60 \sin 120^\circ}{87} = \frac{60(0.8660)}{87} = 0.5972$$

$$\angle AOR = 37^\circ \quad \text{(to the nearest degree)}$$

$$\angle AOE = 180^\circ - 37^\circ = 143^\circ, \text{ and the required}$$

force is a force of 87 pounds, 143° from the 40-pound force in the opposite direction from the 60-pound force.

CHAPTER 6

POWERS AND ROOTS

BASE, EXPONENT, POWER

• PROBLEM 6-1

Simplify: (a) 3^{-2} (b) $\frac{1}{5^{-2}}$

Solution: (a) Since $x^{-a} = \frac{1}{x^a}$, $3^{-2} = \frac{1}{3^2} = \frac{1}{3 \cdot 3} = \frac{1}{9}$.

(b) Again, recall $\frac{1}{x^a} = x^{-a}$; hence,

$$\frac{1}{5^{-2}} = 5^{-(-2)} = 5^2 = 5 \cdot 5 = 25.$$

• PROBLEM 6-2

Simplify the following expressions:

(a) -3^{-2} (b) $(-3)^{-2}$ (c) $\frac{-3}{4^{-1}}$

Solution:

(a) Here the exponent applies only to 3.
Since $x^{-y} = \frac{1}{x^y}$, $-3^{-2} = -(3^{-2}) = -\frac{1}{3^2} = -\frac{1}{9}$.

(b) In this case the exponent applies to the negative base.
Thus, $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{(-3)(-3)} = \frac{1}{9}$.

(c) $\frac{-3}{4^{-1}} = \frac{-3}{(\frac{1}{4})^1} = \frac{-3}{\frac{1^1}{4^1}} = \frac{-3}{\frac{1}{4}}$.

Division by a fraction is equivalent to multiplication by that fraction's reciprocal, thus

$$\frac{-3}{\frac{1}{4}} = -3 \cdot \frac{4}{1} = -12,$$

and $$\frac{-3}{4^{-1}} = -12.$$

• PROBLEM 6-3

Evaluate:

(a) $8\left(-\frac{1}{4}\right)^0$ (b) $6^0 + (-6)^0$ (c) $-7(-3)^0$ (d) 9^{-1} (e) 7^{-2}.

Solution: Note $x^0 = 1$ and $x^{-a} = \frac{1}{x^a}$ for all non-zero real numbers x,

(a) $8\left(-\frac{1}{4}\right)^0 = 8(1) = 8$

(b) $6^0 + (-6)^0 = 1 + 1 = 2$

(c) $-7(-3)^0 = -7(1) = -7$

(d) $9^{-1} = \frac{1}{9^1} = \frac{1}{9}$

(e) $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$

• PROBLEM 6-4

Use the theorems on exponents to perform the indicated operations:

(a) $5x^5 \cdot 2x^2$ (b) $\left(x^4\right)^6$ (c) $\frac{8y^8}{2y^2}$ (d) $\frac{x^3}{x^6}\left(\frac{7}{x}\right)^2$.

Solution: Noting the following properties of exponents:

(1) $a^b \cdot a^c = a^{b+c}$ (2) $\left(a^b\right)^c = a^{b \cdot c}$ (3) $\frac{a^b}{a^c} = a^{b-c}$ (4) $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

we proceed to evaluate these expressions.

(a) $5x^5 \cdot 2x^2 = 5 \cdot 2 \cdot x^5\ x^2 = 10 \cdot x^5 \cdot x^2 = 10x^{5+2} = 10x^7$

(b) $\left(x^4\right)^6 \quad = x^{4 \cdot 6} \qquad = x^{24}$

(c) $\dfrac{8y^8}{2y^2} \quad = \dfrac{8}{2} \cdot \dfrac{y^8}{y^2} \quad = 4 \cdot y^{8-2} \quad = 4y^6$

(d) $\left(\dfrac{x^3}{x^6}\right)\left(\dfrac{7}{x}\right)^2 = \left(\dfrac{x^3}{x^6}\right)\left(\dfrac{7^2}{x^2}\right) \quad = \dfrac{x^3 \cdot 49}{x^6 \cdot x^2} \quad = \dfrac{49x^3}{x^{6+2}} = \dfrac{49x^3}{x^8} = \dfrac{49x^3}{x^{5+3}}$

$$= \frac{49\cancel{x^3}}{x^5 \cdot \cancel{x^3}} \qquad = \frac{49}{x^5}$$

• PROBLEM 6-5

Write $5x^{-3}y^0$ without zero or negative exponents.

Solution: Since $a^{-b} = \dfrac{1}{a^b}$ by definition,

$$x^{-3} = \frac{1}{x^3}$$

and since $x^0 = 1$ by definition (any real non-zero base raised to an exponent of zero equals one),

$$y^0 = 1 \text{ .}$$

Substituting these values for x^{-3} and y^0 we obtain

$$5x^{-3}y^0 = 5 \cdot \frac{1}{x^3} \cdot 1 = \frac{5}{x^3} \text{ .}$$

• PROBLEM 6-6

Simplify the quotient $\dfrac{2x^0}{(2x)^0}$.

Solution: The following two laws of exponents can be used to simplify the given quotient:

1) $a^0 = 1$ where a is any non-zero real number, and

2) $(ab)^n = a^nb^n$ where a and b are any two numbers.

In the given quotient, notice that the exponent in the numerator applies only to the letter x. However, the exponent in the denominator applies to both the number 2 and the letter x; that is, the exponent in the denominator applies to the entire term (2x). Using the first law, the numerator can be rewritten as:

$$2x^0 = 2(1) = 2$$

Using the second law with n = 0, the denominator can be rewritten as:

$$(2x)^0 = 2^0x^0$$

Using the first law again to further simplify the denominator:

$$(2x)^0 = 2^0x^0$$
$$= (1)(1)$$
$$= 1$$

Therefore,

$$\frac{2x^0}{(2x)^0} = \frac{2}{1} = 2$$

• PROBLEM 6-7

Express $\left[\dfrac{a^{-2}}{b^{-3}}\right]^{-2}$ using only positive exponents.

Solution A: By the law of exponents which states that $(x)^{-n} = \dfrac{1}{x^n}$ where n is a positive integer,

$$\left[\frac{a^{-2}}{b^{-3}}\right]^{-2} = \frac{1}{\left[\dfrac{a^{-2}}{b^{-3}}\right]^2}.$$

Since $\left[\dfrac{x}{y}\right]^n = \dfrac{x^n}{y^n}$, $\left[\dfrac{a^{-2}}{b^{-3}}\right]^2 = \dfrac{(a^{-2})^2}{(b^{-3})^2}$. Also, since $(x^m)^n = x^{mn}$,

$(a^{-2})^2 = a^{(-2)(2)} = a^{-4}$, $(b^{-3})^2 = b^{(-3)(2)} = b^{-6}$. Hence,

$$\left[\frac{a^{-2}}{b^{-3}}\right]^{-2} = \frac{1}{\left[\dfrac{a^{-2}}{b^{-3}}\right]^2}$$

$$= \frac{1}{\dfrac{a^{-4}}{b^{-6}}}$$

$$= \frac{1}{\dfrac{(a^4)^{-1}}{(b^6)^{-1}}}$$

$$= \frac{1}{\left[\dfrac{a^4}{b^6}\right]^{-1}}$$

$$\frac{1}{\left[\dfrac{1}{\dfrac{a^4}{b^6}}\right]}$$

Note that division is the same as multiplying the numerator by the reciprocal of the denominator. This principle is applied to the term in brackets.

$$\left[\frac{a^{-2}}{b^{-3}}\right]^{-2} = \frac{1}{(1)\left[\dfrac{b^6}{a^4}\right]} = \left(\frac{1}{\dfrac{b^6}{a^4}}\right)$$

Applying the same principle to the term in parenthesis on the right side of the equation:

$$\left[\frac{a^{-2}}{b^{-3}}\right]^{-2} = \left[\frac{1}{\dfrac{b^6}{a^4}}\right] = (1)\left(\frac{a^4}{b^6}\right) = \frac{a^4}{b^6}.$$

Solution B: Since $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$, $\left[\frac{a^{-2}}{b^{-3}}\right]^{-2} = \frac{(a^{-2})^{-2}}{(b^{-3})^{-2}}$. Also, since $(x^m)^n = x^{mn}$, $(a^{-2})^{-2} = a^{(-2)(-2)} = a^4$, and $(b^{-3})^{-2} = b^{(-3)(-2)} = b^6$. Hence,

$$\left[\frac{a^{-2}}{b^{-3}}\right]^{-2} = \frac{a^4}{b^6}.$$

• PROBLEM 6-8

Evaluate the following expression: $\dfrac{12x^7y}{3x^2y^3}$

Solution: Noting (1) $\frac{abc}{def} = \frac{a}{d}\cdot\frac{b}{e}\cdot\frac{c}{f}$, (2) $a^{-b} = \frac{1}{a^b}$ and (3) $\frac{a^b}{a^c} = a^{b-c}$ for all non-zero real values of a,d,e,f,we

• PROBLEM 6-9

Evaluate $\sqrt{400}$.

Solution: $400 = 4 \times 100$

Thus, $\sqrt{400} = \sqrt{4 \times 100}$

Since $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$,

$$\sqrt{400} = \sqrt{4}\,\sqrt{100} = 2 \cdot 10 = 20$$

Check: If $\sqrt{400}$ is 20, then 20^2 must equal 400, which is true. Hence, 20 is the solution.

• PROBLEM 6-10

Evaluate $16^{-\frac{3}{4}}$.

Solution:

$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{(\sqrt[4]{16})^3}.$$

Note that $2^4 = 2\cdot2\cdot2\cdot2 = 16$, hence $\sqrt[4]{16} = 2$. Thus, $16^{-\frac{3}{4}} = \frac{1}{2^3} = \frac{1}{2\cdot2\cdot2} = \frac{1}{8}$.

• PROBLEM 6-11

Find the numerical value of each of the following.
(a) $8^{2/3}$ (b) $25^{3/2}$

Solution:

(a) Since $x^{a/b} = \left(x^{1/b}\right)^a$, $8^{2/3} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$

(b) Similarly, $25^{3/2} = \left(25^{1/2}\right)^3 = 5^3 = 125$.

• **PROBLEM 6-12**

Simplify $\sqrt{12} - \sqrt{27}$.

Solution: Here we have two different radicals, yet when each is simplified, the distributive law gives a simpler form for the expression. Note that 12 and 27 both have a factor 3, hence

$$\sqrt{12} - \sqrt{27} = \sqrt{4 \cdot 3} - \sqrt{9 \cdot 3}$$

Because $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, $= \sqrt{4} \cdot \sqrt{3} - \sqrt{9} \cdot \sqrt{3}$

$$= 2\sqrt{3} - 3\sqrt{3}$$

Now, we use the distributive law, $= (2 - 3)\sqrt{3}$

$$= (-1)\sqrt{3}$$

$$= -\sqrt{3}$$

• **PROBLEM 6-13**

If a = 3 and b = 2, find $(6a - b)^{-5/4}$.

Solution: Substitute a = 3 and b = 2: $(6 \cdot 3 - 2)^{-5/4}$

Perform the indicated multiplication: $(18-2)^{-5/4}$

$$= 16^{-5/4}$$

Since $x^{-y} = \frac{1}{x^y}$: $= \frac{1}{16^{5/4}}$

$$= \frac{1}{\left(\sqrt[4]{16}\right)^5}$$

Since $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$,

$\sqrt[4]{16} = 2$. Hence: $= \frac{1}{2^5}$

$$= \frac{1}{32}$$

• **PROBLEM 6-14**

Find the product $\sqrt[4]{x^3y} \cdot \sqrt[4]{xy^2}$ and simplify.

Solution: Note that $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{ab}$; thus,

$$\sqrt[4]{x^3y} \cdot \sqrt[4]{xy^2} = \sqrt[4]{(x^3y)(xy^2)}.$$

Recall that when multiplying, we add exponents; hence

$$\left(x^3y^1\right)\left(x^1y^2\right) = \left(x^{3+1}\ y^{1+2}\right), \text{ and}$$

we obtain,
$$= \sqrt[4]{x^4y^3}$$
$$= \sqrt[4]{x^4}\left(\sqrt[4]{y^3}\right)$$

Now, since $\sqrt[4]{x^4} = \left(x^{\frac{1}{4}}\right)^4 = x^1 = x$, $\sqrt[4]{x^3y} \cdot \sqrt[4]{xy^2} = x\ \sqrt[4]{y^3}$.

• **PROBLEM 6-15**

Rationalize $\dfrac{\sqrt{3xy}}{\sqrt{2x} - \sqrt{3y}}$.

Solution: To rationalize a fraction, we multiply numerator and denominator by the conjugate of the denominator (where the conjugate of a + b is a - b). In our example, we multiply numerator and denominator by the conjugate of $\sqrt{2x} - \sqrt{3y}$, which is $\sqrt{2x} + \sqrt{3y}$. Thus,

$$\frac{\sqrt{3xy}}{\sqrt{2x} - \sqrt{3y}} = \frac{\sqrt{3xy}}{\sqrt{2x} - \sqrt{3y}} \cdot \frac{\sqrt{2x} + \sqrt{3y}}{\sqrt{2x} + \sqrt{3y}}$$

$$= \frac{\sqrt{3xy}\ (\sqrt{2x} + \sqrt{3y})}{(\sqrt{2x})^2 - (\sqrt{3y})^2}.$$

Since $(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$,

$$(\sqrt{2x})^2 = 2x$$

$$(\sqrt{3y})^2 = 3y.$$

Making these substitutions,

$$\frac{\sqrt{3xy}\ (\sqrt{2x} + \sqrt{3y})}{(\sqrt{2x})^2 - (\sqrt{3y})^2} = \frac{\sqrt{3xy}\ (\sqrt{2x} + \sqrt{3y})}{2x - 3y}$$

$$= \frac{\sqrt{3xy} \cdot \sqrt{2x} + \sqrt{3xy} \cdot \sqrt{3y}}{2x - 3y}$$

Since $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$ and $\sqrt{a}\ \sqrt{b}\ \sqrt{c} = \sqrt{abc}$,

$$\frac{\sqrt{3xy} \cdot \sqrt{2x} + \sqrt{3xy} \cdot \sqrt{3y}}{2x - 3y} = \frac{\sqrt{6x^2y} + \sqrt{9xy^2}}{2x - 3y}$$

$$= \frac{\sqrt{x^2}\ \sqrt{6y} + \sqrt{9}\ \sqrt{y^2}\ \sqrt{x}}{2x - 3y}$$

$$= \frac{x\sqrt{6y} + 3y\sqrt{x}}{2x - 3y}$$

• PROBLEM 6-16

Simplify $(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{2} - \sqrt{6})$.

Solution: Using the distributive property the expression on the right can be multiplied by each term in the expression on the left or vice versa. Hence,

$$(\sqrt{3} + \sqrt{2})(\sqrt{2} - \sqrt{6}) = \sqrt{3}(\sqrt{2} - \sqrt{6}) + \sqrt{2}(\sqrt{2} - \sqrt{6})$$

The distributive property again enables us to multiply the terms on the left by their respective right hand members:

$$= (\sqrt{3} \cdot \sqrt{2}) - (\sqrt{3} \cdot \sqrt{6}) + (\sqrt{2} \cdot \sqrt{2}) - (\sqrt{2} \cdot \sqrt{6})$$

Then since,

$$\sqrt{a} \cdot \sqrt{6} = \sqrt{ab},$$

$$(\sqrt{3} + 2)(\sqrt{2} - \sqrt{6}) = (\sqrt{6}) - (\sqrt{18}) + (2) - (\sqrt{12})$$

$$= \sqrt{6} - 3\sqrt{2} + 2 - 2\sqrt{3}.$$

SCIENTIFIC NOTATION

• PROBLEM 6-17

Use scientific notation to express each number.
(a) 4,375 (b) 186,000 (c) 0.00012 (d) 4,005

Solution: A number expressed in scientific notation is written as a product of a number between 1 and 10 and a power of 10. The number between 1 and 10 is obtained by moving the decimal point of the number (actual or implied) the required number of digits. The power of 10, for a number greater than 1, is positive and is one less than the number of digits before the decimal point in the original number. The power of 10, for a number less than 1, is negative and is one more than the number of zeros immediately following the decimal point in the original number. Hence,

(a) $4,375 = 4.375 \times 10^3$ (b) $186,000 = 1.86 \times 10^5$

(c) $0.00012 = 1.2 \times 10^{-4}$ (d) $4,005 = 4.005 \times 10^3$

• PROBLEM 6-18

Express $\frac{6,400,000}{400}$ in scientific notation.

Solution: In order to solve this problem, we express the numerator and denominator as the product of a number between 1 and 10 and a power of 10. This is known as scientific notation. Thus

$$6,400,000 = 6.4 \times 1,000,000 = 6.4 \times 10^6$$
$$400 = 4 \times 100 = 4 \times 10^2$$

Thus,

$$\frac{6,400,000}{400} = \frac{6.4 \times 10^6}{4.0 \times 10^2}$$

Since $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{c}$: $= \frac{6.4}{4.0} \times \frac{10^6}{10^2}$

Since $\frac{a^x}{a^y} = a^{x-y}$: $= 1.6 \times 10^4$

CHAPTER 7

FUNCTIONS AND LINEAR RELATIONSHIPS

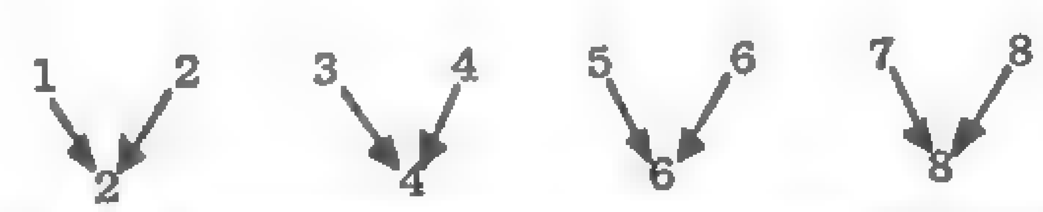
FUNCTIONS AND RELATIONS

• PROBLEM 7-1

If $X = \{1,2,3,4,5,6,7,8\}$ and $Y = \{2,4,6,8\}$, use $f(x)$ notation to indicate the image of each element of X in the following mapping.

Set X

Set Y



Solution: The mapping of an element x in the set X to an element y in the set Y may be written in $f(x)$ notation as $f(x) = y$ when f is a function mapping x to y. That is, $f: x \to y$. Therefore,

$$f(1) = 2,\ f(2) = 2,\ f(3) = 4,\ f(4) = 4,\ f(5) = 6,\ f(6) = 6,$$
$$f(7) = 8,\ f(8) = 8$$

• PROBLEM 7-2

Find the image of each element in

$$A = \{1,2,3,4,5,6,7,8,9\}$$

under the following mapping:

$$f(x) = \begin{cases} 2x, & \text{if } x < 5 \\ 8, & \text{if } x \ge 5 \end{cases}$$

Solution: The image of each element in $A = \{1,2,3,4,5,6,7,8,9\}$ under the mapping $f(x)$, is $f(1), f(2), f(3), f(4), f(5), f(6), f(7), f(8), f(9)$. $f(x)$ has two corresponding values, depending on the value of

x. If $x < 5$, $f(x) = 2x$, thus for

$x = 1,\ f(1) = 2(1) = 2$
$x = 2,\ f(2) = 2(2) = 4$
$x = 3,\ f(3) = 2(3) = 6$
$x = 4,\ f(4) = 2(4) = 8$

and if $x \ge 5,\ f(x) = 8$, thus for

$x = 5,\ f(5) = 8$
$x = 6,\ f(6) = 8$
$x = 7,\ f(7) = 8$
$x = 8,\ f(8) = 8$
$x = 9,\ f(9) = 8$

• PROBLEM 7-3

Which of the following sets are functions of x?

$A = \{(5,1),(4,2),(4,3),(6,4)\}$,

$B = \{(x,y) \mid y = |x|\}$,

$C = \{(x,y) \mid x = |y|\}$?

Solution: A function is a relation having the property that each member of its domain is paired with exactly one member of its range. Thus, set A is not a function, for it contains the pairs (4,2) and (4,3) - that is, one member of its domain, 4, is paired with more than one member of its range, 2 and 3. If each x value has only one corresponding y value, any vertical line only intersects the graph of a function at one point. Thus, from

Fig. 1 $y=|x|$

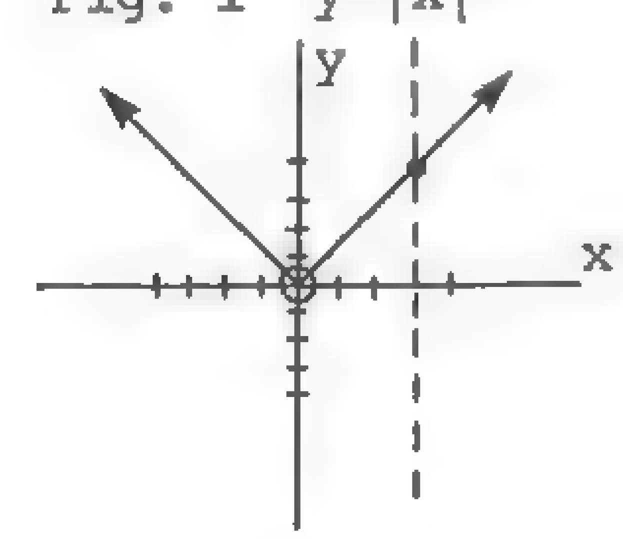


Fig. 2 $x=|y|$

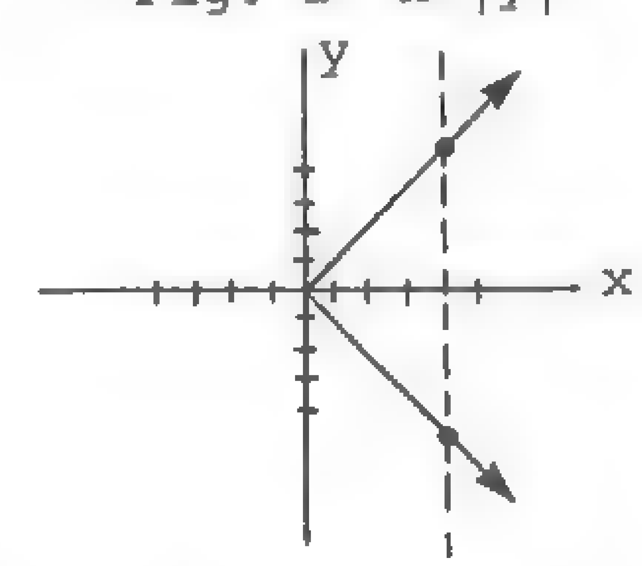


figure 1 we note that set B is a function. Notice that a function may contain two pairs with the same second member; for example, our function B contains the pairs (1,1) and (- 1,1).

From figure 2 we note that a vertical line intersects the graph of C in two places, thus there are x

values of C which have more than one corresponding y value, and C is not a function.

• PROBLEM 7-4

Let the domain of $M = \{(x,y): y = x\}$ be the set of real numbers. Is M a function?

Fig. A

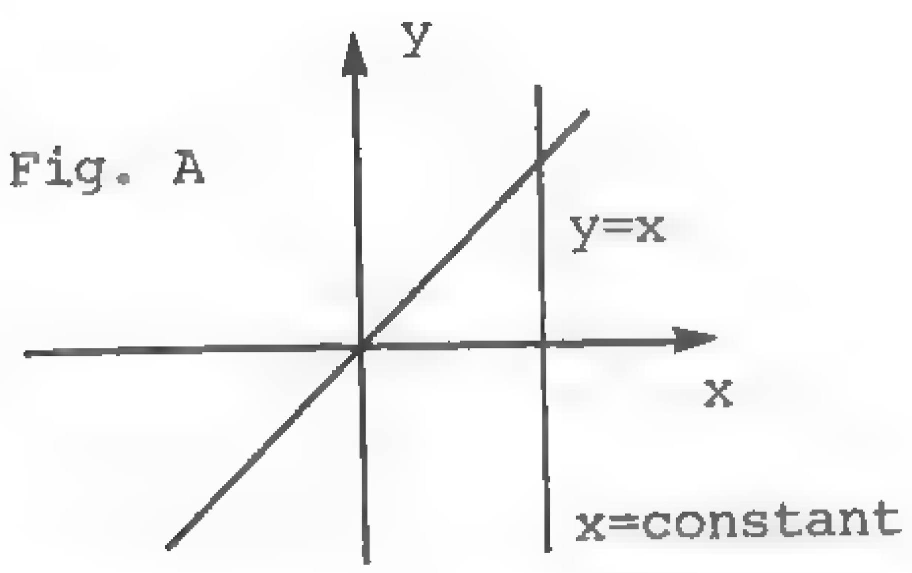
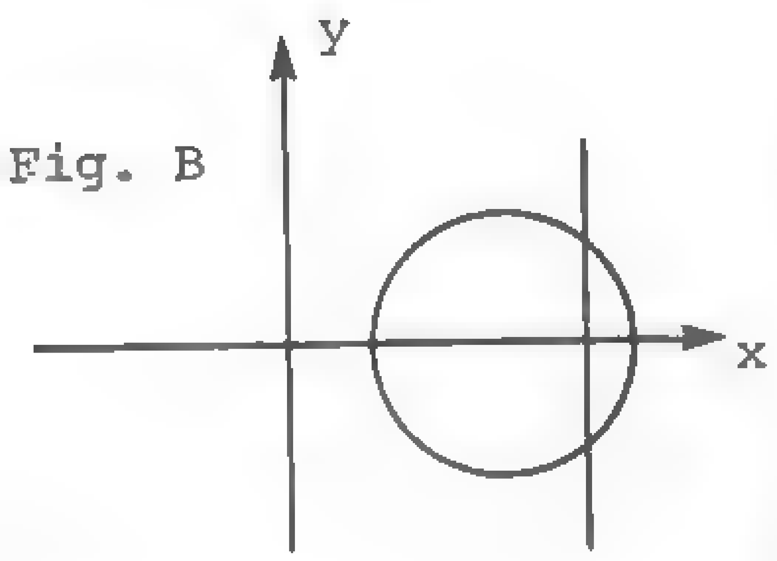


Fig. B



Solution: The range is also the set of real numbers since $y = \{y | y = x\}$. The graph of $y = x$ is the graph of a line ($y = mx + b$ where $m = 1$ and $b = 0$). See fig. A. If for every value of x in the domain, there corresponds only one y value then y is said to be a function of x. Since each element in the domain of M has exactly one element for its image, M is a function. Also notice that a vertical line (x = constant) crosses the graph $y = x$ only once. Whenever this is true the graph defines a function. Consult figure B.
The vertical line (x = constant) crosses the graph of the circle twice; i.e., for each x,y is not unique, therefore the graph does not define a function.

• PROBLEM 7-5

If $g(x) = x^2 + 5x - 3$, find $g(-7)$.

Solution: Substitute -7 for x everywhere in the equation:

$$g(-7) = (-7)^2 + 5(-7) - 3$$
$$= 49 - 35 - 3 = 11.$$

• PROBLEM 7-6

If $f(t) = 6t + 13$, find $f(5) - f(4)$.

Solution: To find f(5) we substitute 5 for t everywhere in the equation, that is:

$$f(5) = 6(5) + 13 = 43$$

Similarly, $f(4) = 6(4) + 13 = 37$
and now subtract f(4) from f(5). Therefore,

$$f(5) - f(4) = 43 - 37 = 6.$$

• PROBLEM 7-7

If $f(x) = (x - 2)/(x + 1)$, find the function values $f(2)$, $f(\frac{1}{2})$, and $f(-3/4)$.

Solution: To find f(2), we replace x by 2 in the given formula for f(x), $f(x) = x - 2/x + 1$; thus

$$f(2) = \frac{2-2}{2+1} = \frac{0}{3} = 0.$$

Similarly, $f(\frac{1}{2}) = \dfrac{\frac{1}{2} - 2}{\frac{1}{2} + 1}$.

Multiply numerator and denominator by 2,

$$= \frac{2(\frac{1}{2} - 2)}{2(\frac{1}{2} + 1)}.$$

Distribute,
$$= \frac{2(\frac{1}{2}) - 2 \cdot 2}{2(\frac{1}{2}) + 2}$$
$$= \frac{1-4}{1+2}$$
$$= -\frac{3}{3} = -1.$$

$$f(-3/4) = \frac{-3/4 - 2}{-3/4 + 1}.$$

Multiply numerator and denominator by 4,

$$= \frac{4(-3/4 - 2)}{4(-3/4 + 1)}.$$

Distribute,
$$= \frac{4(-3/4) - 4(2)}{4(-3/4) + 4(1)}$$
$$= \frac{-3-8}{-3+4}$$
$$= \frac{-11}{1}$$
$$= -11.$$

• PROBLEM 7-8

If $g(x) = x^2 - 2x + 1$, find the given element in the range.

a) $g(-2)$ b) $g(0)$ c) $g(a + 1)$ d) $g(a - 1)$

Solution: a) To find g(-2), substitute -2 for x in the given equation.

$$g(x) = g(-2)$$
$$= (-2)^2 - 2(-2) + 1$$
$$= 4 + 4 + 1$$
$$= 8 + 1$$
$$= 9$$

Hence, $g(-2) = 9$

b) To find g(0), substitute 0 for x in the given equation.

$$g(x) = g(0)$$
$$= (0)^2 - 2(0) + 1$$
$$= 0 - 0 + 1$$
$$= 1$$

Hence, $g(0) = 1$

c) To find g(a + 1), substitute a + 1 for x in given equation.

$$g(x) = g(a + 1)$$
$$= (a + 1)^2 - 2(a + 1) + 1$$
$$= (a^2 + 2a + 1) - 2a - 2 + 1$$
$$= a^2 + \cancel{2a} + 1 - \cancel{2a} - 2 + 1$$
$$= a^2 + 1 - 2 + 1$$
$$= a^2 + 0$$
$$= a^2$$

Hence, $g(a + 1) = a^2$.

d) To find g(a - 1), substitute a - 1 for x in given equation.

$$g(x) = g(a - 1)$$

$$= (a - 1)^2 - 2(a - 1) + 1$$
$$= (a^2 - 2a + 1) - 2a + 2 + 1$$
$$= a^2 - 2a + 1 - 2a + 2 + 1$$
$$= a^2 - 4a + 4$$

Hence, $g(a - 1) = a^2 - 4a + 4$

• PROBLEM 7-9

Find the domain D and the range R of the function $\left(x, \frac{x}{|x|}\right)$.

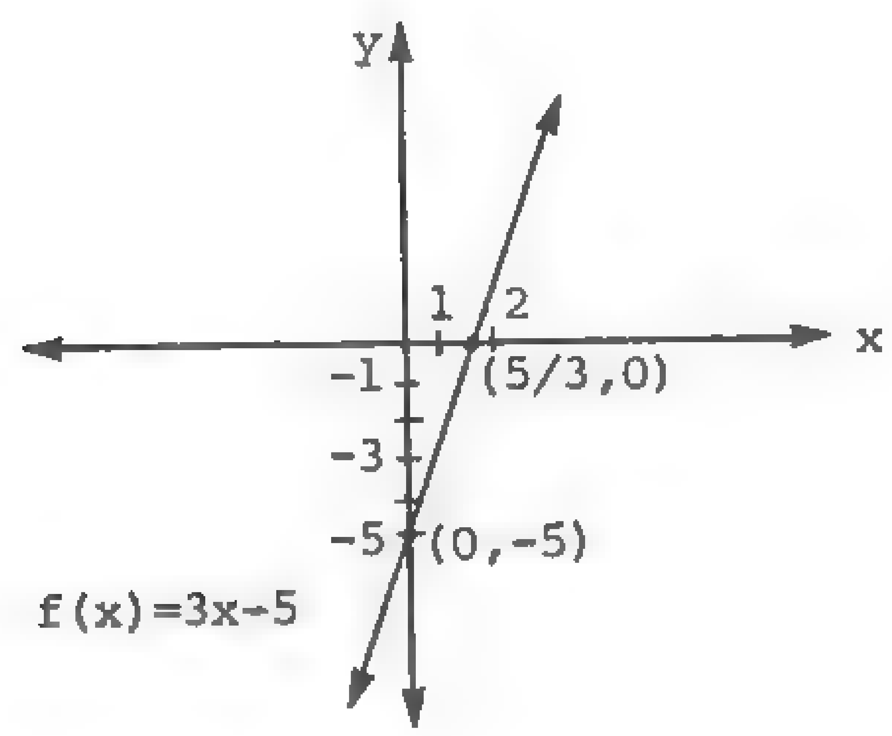
Solution: Note that the y-value of any coordinate pair (x,y) is $\frac{x}{|x|}$. We can replace x in the formula $\frac{x}{|x|}$ with any number except 0, since the denominator, $|x|$, can not equal 0, (i.e. $|x| \neq 0$) which is equivalent to $x \neq 0$. This is because division by 0 is undefined. Therefore, the domain D is the set of all real numbers except 0. If x is negative, i.e. $x < 0$, then $|x| = -x$ by definition. Hence, if x is negative, then $\frac{x}{|x|} = \frac{x}{-x} = -1$. If x is positive, i.e. $x > 0$, then $|x| = x$ by definition. Hence, if x is positive, then $\frac{x}{|x|} = \frac{x}{x} = 1$. (The case where x = 0 has already been found to be undefined). Thus, there are only two numbers -1 and 1 in the range R of the function; that is, $R = \{-1, 1\}$.

• PROBLEM 7-10

Find the zeros of the function f if $f(x) = 3x - 5$.

Solution: The zeros of the function $f(x) = 3x - 5$ are those values of x for which $3x - 5 = 0$:

$$3x - 5 = 0$$
$$3x = 5$$
$$x = \frac{5}{3}$$



Thus $x = 5/3$ is a zero of $f(x) = 3x - 5$, which means that the graph of $f(x)$ crosses the x axis at the point $(5/3, 0)$ (see figure).

• PROBLEM 7-11

Find the zeros of the function

$$\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10}.$$

Solution: Let the function $f(x)$ be equal to $\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10}$. A number, a, is a zero of a function $f(x)$ if $f(a) = 0$. A zero of $f(x)$ is a root of the equation $f(x) = 0$. Thus, the zeros of the function are the roots of the equation

$$\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10} = 0.$$

The least common denominator, LCD, of the denominators of 5, 4, and 10 is 20. This is a fractional equation which can be solved by multiplying both members of the equation by the LCD.

$$20\left(\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10}\right) = (20)(0)$$

$$4(2x+7) + 5(3x-5) + (2\cdot 33) = 0.$$

Distributing,

$$\begin{aligned} 8x + 28 + 15x - 25 + 66 &= 0. \\ 23x + 69 &= 0 \\ 23x &= -69 \\ x &= -3 \end{aligned}$$

Hence $x = -3$ is the zero of the given function.

SOLVING LINEAR EQUATIONS

• PROBLEM 7-12

Solve the equation $6x - 3 = 7 + 5x$.

Solution: To solve for x in the equation $6x - 3 = 7 + 5x$, we wish to obtain an equivalent equation in which each term in one member involves x, and each term in the other member is a constant. If we add $(-5x)$ to both members, then only one side of the equation will have an x term:

$$6x - 3 + (-5x) = 7 + 5x + (-5x)$$

$$6x + (-5x) - 3 = 7 + 0$$

$$x - 3 = 7$$

Now, adding 3 to both sides of the equation we obtain,

$$x - 3 + 3 = 7 + 3$$

$$x + 0 = 10$$

$$x = 10$$

Thus, our solution is $x = 10$. Now we check this value.

Check: Substitute 10 for x in the original equation:

$$6x - 3 = 7 + 5x$$

$$6(10) - 3 = 7 + 5(10)$$

$$60 - 3 = 7 + 50$$

$$57 = 57.$$

• PROBLEM 7-13

Solve for x: $2x + 5 = 7 - x$.

Solution: Add x to both sides: $2x + 5 + x = 7 - x + x$

Combine terms: $3x + 5 = 7$

Subtract 5 from both sides: $3x + 5 - 5 = 7 - 5$

Combine terms: $3x = 2$

Divide both sides by 3: $\frac{3x}{3} = \frac{2}{3}$

$$x = \frac{2}{3}$$

• PROBLEM 7-14

Solve the equation $2(x + 3) = (3x + 5) - (x - 5)$.

Solution: We transform the given equation to an equivalent equation where we can easily recognize the solution set.

$$2(x + 3) = 3x + 5 - (x - 5)$$

Distribute, $2x + 6 = 3x + 5 - x + 5$

Combine terms, $2x + 6 = 2x + 10$

Subtract $2x$ from both sides, $6 = 10$

Since $6 = 10$ is not a true statement, there is no real number which will make the original equation true. The equation is inconsistent and the solution set is ϕ, the empty set.

• PROBLEM 7-15

Solve the equation $\frac{3}{4}x + \frac{7}{8} + 1 = 0$.

Solution: There are several ways to proceed. First we observe that $\frac{3}{4}x + \frac{7}{8} + 1 = 0$ is equivalent to

$\frac{3}{4}x + \frac{7}{8} + \frac{8}{8} = 0$, where we have converted 1 into $\frac{8}{8}$. Now, combining fractions we obtain:

$$\frac{3}{4}x + \frac{15}{8} = 0$$

Subtract $\frac{15}{8}$ from both sides:

$$\frac{3}{4}x = \frac{-15}{8}$$

Multiplying both sides by $\frac{4}{3}$:

$$\left(\frac{4}{3}\right)\frac{3}{4}x = \left(\frac{4}{3}\right)\left(\frac{-15}{8}\right)$$

Cancelling like terms in numerator and denominator:

$$x = \frac{-5}{2}$$

A second method is to multiply both sides of the equation by the least common denominator, 8:

$$8\left(\frac{3}{4}x + \frac{7}{8} + 1\right) = 8(0)$$

Distributing: $8\left(\frac{3}{4}\right)x + 8\left(\frac{7}{8}\right) + 8 \cdot 1 = 0$

$$(2 \cdot 3)x + 7 + 8 = 0$$

$$6x + 15 = 0$$

Subtract 15 from both sides: $6x = -15$

Divide both sides by 6: $x = \frac{-15}{6}$

Cancelling 3 from numerator and denominator: $x = \frac{-5}{2}$

• PROBLEM 7-16

Solve for x:

$$\frac{x}{2} + \frac{x}{3} = 12.$$

Solution: The Least Common Denominator is 6. Multiply both members of the equation by 6: $6\left(\frac{x}{2} + \frac{x}{3}\right) = 6(12)$.

Use distributive law: $3x + 2x = 72$.

Collect terms: $5x = 72$.

Divide by 5: Therefore, $x = 14\frac{2}{5}$.

Check: Substitute $14\frac{2}{5} = \frac{72}{5}$ for x in the given equation:

$$\frac{\frac{72}{5}}{2} + \frac{\frac{72}{5}}{3} = 12$$

$$\left(\frac{72}{5} \cdot \frac{1}{2}\right) + \left(\frac{72}{5} \cdot \frac{1}{3}\right) = 12$$

$$\frac{36}{5} + \frac{24}{5} = 12$$

$$\frac{60}{5} = 12$$

$$12 = 12$$

• PROBLEM 7-17

Solve $\frac{1}{2x} - \frac{5}{16} = \frac{1}{x}$.

Solution: In order to rid an equation of fractions we multiply both sides by the least common multiple. In this case our L.C.M. is 16x:

$$16x\left(\frac{1}{2x} - \frac{5}{16}\right) = 16x\left(\frac{1}{x}\right)$$

Distributing, $16x\left(\frac{1}{2x}\right) - 16x\left(\frac{5}{16}\right) = \frac{16x}{x}$

Cancelling out like terms in numerator and denominator:

$$8 - 5x = 16$$

Subtracting 8 from both sides:

$$-5x = 16 - 8$$

$$-5x = 8$$

Dividing both sides by -5:

$$x = -\frac{8}{5}$$

Check:

Substitute $\frac{-8}{5}$ for x in $\frac{1}{2x} - \frac{5}{16} = \frac{1}{x}$:

$$\frac{1}{2\left(-\frac{8}{5}\right)} - \frac{5}{16} = \frac{1}{\left(-\frac{8}{5}\right)}$$

$$\frac{1}{\left(\frac{-16}{5}\right)} - \frac{5}{16} = \frac{1}{-\frac{8}{5}}$$

Since division by a fraction is equivalent to multiplication by the reciprocal

$$\frac{1}{\left(\frac{-16}{5}\right)} = 1 \cdot \left(\frac{5}{-16}\right) = \frac{-5}{16}$$

and $\frac{1}{\left(\frac{-8}{5}\right)} = 1 \cdot \left(\frac{5}{-8}\right) = \frac{-5}{8}$

Hence, $\frac{-5}{16} - \frac{5}{16} = \frac{-5}{8}$

$$\frac{-10}{16} = \frac{-5}{8}$$

$$\frac{-5}{8} = \frac{-5}{8}$$

• PROBLEM 7-18

Solve $\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$ for a.

Solution: To solve for a we must obtain a alone on one side of the equation,

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b}. \quad (1)$$

To do this we proceed as follows: Multiply Equation (1) by Rab. Then,

$$Rab\left(\frac{1}{R}\right) = Rab\left(\frac{1}{a} + \frac{1}{b}\right).$$

Therefore: $\frac{\cancel{R}ab}{\cancel{R}} = \frac{R\cancel{a}b}{\cancel{a}} + \frac{Ra\cancel{b}}{\cancel{b}}.$

Therefore: $ab = Rb + Ra.$ (2)

Subtracting Ra from both sides of Equation (2), we obtain:

$$ab - Ra = Rb + Ra - Ra.$$

Therefore: $ab - Ra = Rb.$ (3)

We can now factor a from both terms of the left side of Equation (3), obtaining:

$$a(b - R) = Rb. \quad (4)$$

Now, we divide both sides of Equation (4) by $(b - R)$:

$$\frac{a\cancel{(b - R)}}{\cancel{(b - R)}} = \frac{Rb}{(b - R)}.$$

Thus, we find $a = \frac{Rb}{b - R}$.

• PROBLEM 7-19

Find the solutions of the equation $\frac{4x - 7}{x - 2} = 3 + \frac{1}{x - 2}$.

Solution: Assume that there is a number x such that

$$\frac{4x - 7}{x - 2} = 3 + \frac{1}{x - 2}$$

In order to eliminate the fractions multiply both sides of the equation by $x - 2$ to obtain

$$(x - 2)\ \frac{4x - 7}{x - 2} = \left(3 + \frac{1}{x - 2}\right)(x - 2)$$

Thus

$$4x - 7 = 3(x - 2) + \frac{x-2}{x-2}$$

$$4x - 7 = 3(x - 2) + 1$$

$$4x - 7 = 3x - 6 + 1$$

$$4x - 7 = 3x - 5$$

Add $(-3x)$ to both sides, $\quad 4x - 7 + (-3x) = -5$

$$x - 7 = -5$$

Add 7 to both sides, $\quad x = -5 + 7$

and hence $x = 2$.

We have shown that if x is a solution of the equation

$$\frac{4x-7}{x-2} = 3 + \frac{1}{x-2},$$

then $x = 2$. But if we substitute $x = 2$ in the right-hand member of the equation we obtain

$$3 + \frac{1}{0}$$

and we know that we cannot divide by zero. Hence 2 is not a solution.

Before we analyze the process which led to the conclusions that 2 was a possible solution to our equation, let us see exactly why our equation has no solution. To do this, we note that

$$3 + \frac{1}{x-2} = 3 \cdot \frac{x-2}{x-2} + \frac{1}{x-2} = \frac{3(x-2)+1}{x-2} = \frac{3x-6+1}{x-2} = \frac{3x-5}{x-2}$$

and hence that the original equation is equivalent to

$$\frac{4x-7}{x-2} = \frac{3x-5}{x-2} \qquad (1)$$

Now we know that two fractions, $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $ad = bc$. Thus (1) holds, providing that $x \neq 2$, if and only if

$$(x - 2)(4x - 7) = (x - 2)(3x - 5) \qquad (2)$$

holds. But, since $x \neq 2$, $x - 2 \neq 0$, and we can divide both sides of (2) by $x - 2$ and have

$$4x - 7 = 3x - 5$$

which gives $x = 2$, a contradiction. In other words, the only possible solution is a number which we knew in advance could not be a solution, and hence there are no solutions to our given equation.

• PROBLEM 7-20

Solve the equation

$$\frac{x}{x+1} + \frac{5}{8} = \frac{5}{2(x+1)} + \frac{3}{4}$$

Solution: Since $(x + 1)$, 8, $2(x + 1)$, and 4, the denominators of the fractions, are all factors of $8(x + 1)$,

and there is no smaller number which contains $(x + 1)$, 8, $2(x + 1)$, and 4 as factors, $8(x + 1)$ is the least common multiple (LCM). We may therefore multiply both sides of the given equation by $8(x + 1)$ to eliminate the fractions.

$$8(x+1)\left(\frac{x}{x+1} + \frac{5}{8}\right) = 8(x+1)\left(\frac{5}{2(x+1)} + \frac{3}{4}\right).$$

Distribute,

$$8(x+1)\left(\frac{x}{x+1}\right) + 8(x+1)\left(\frac{5}{8}\right)$$

$$= 8(x+1)\left(\frac{5}{2(x+1)}\right) + 8(x+1)\left(\frac{3}{4}\right)$$

Cancel like terms, $\quad 8x + 5(x + 1) = 4(5) + 6(x + 1)$

Distribute, $\quad 8x + 5x + 5 = 20 + 6x + 6$

Combine terms, $\quad 13x + 5 = 26 + 6x$

Add $(-6x)$ to both sides

$$13x + 5 + (-6x) = 26 + 6x + (-6x)$$

$$7x + 5 = 26$$

Add (-5) to both sides

$$7x + 5 + (-5) = 26 + (-5)$$

$$7x = 21$$

Divide both sides by 7, $\quad x = 3$.

Thus, our solution is $x = 3$, and we have

$$\left\{x \,\middle|\, \frac{x}{x+1} + \frac{5}{8} = \frac{5}{2(x+1)} + \frac{3}{4}\right\} = \{3\}.$$

To verify this statement we perform the following check.

Check: Replace x by 3 in the original equation,

$$\frac{x}{x+1} + \frac{5}{8} = \frac{5}{2(x+1)} + \frac{3}{4}$$

$$\frac{3}{3+1} + \frac{5}{8} = \frac{5}{2(3+1)} + \frac{3}{4}$$

$$\frac{3}{4} + \frac{5}{8} = \frac{5}{2(4)} + \frac{3}{4}$$

$$\frac{3}{4} + \frac{5}{8} = \frac{5}{8} + \frac{3}{4}$$

$$\frac{3}{4} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$$

Since substitution of x by 3 results in this equi-

valent equation, which is always true, 3 is indeed a root of the equation.

• PROBLEM 7-21

Solve $\sqrt{x-3} = 4$.

Solution: Square both sides of the given equation to obtain:

$$\left(\sqrt{x-3}\right)^2 = 4^2$$

Note $(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a} = \sqrt{a^2} = a$; thus $\left(\sqrt{x-3}\right)^2 = x - 3$, and we obtain:

$$x - 3 = 16$$

$$x = 19$$

Check: Substitute 19 for x in the original equation,

$$\sqrt{x-3} = 4$$

$$\sqrt{19-3} = 4$$

$$\sqrt{16} = 4$$

$$4 = 4$$

• PROBLEM 7-22

Solve the equation

$$\sqrt{x} = 7 + \sqrt{x-7}$$

Solution: Squaring both sides of the given equation,

$$x = 49 + 14\sqrt{x-7} + x - 7$$

Simplifying

$$-42 = 14\sqrt{x-7}$$

$$-3 = \sqrt{x-7} \qquad (1)$$

Squaring both sides of equation (1),

$$9 = x - 7$$

$$x = 16$$

Checking the root by substitution in the given equation:

$$\sqrt{16} \neq 7 + \sqrt{16-7}$$

$$4 \neq 7 + 3$$

Clearly $x = 16$ does not satisfy the given equation, and therefore the equation has no roots. The fact that the given equation has no roots could have been anticipated from equation (1), $-3 = \sqrt{x-7}$, since the positive root is indicated in the original equation.

• PROBLEM 7-23

Solve $\sqrt{2}x - 2 = 2x - \sqrt{2}$.

Solution: Add $(-2x)$ to both sides of the given equation:

$$\sqrt{2}x - 2 - 2x = -\sqrt{2}$$

Now, add 2 to both sides:

$$\sqrt{2}x - 2x = 2 - \sqrt{2}$$

Use the distributive law:

$$x(\sqrt{2} - 2) = 2 - \sqrt{2}$$

$$x = \frac{2 - \sqrt{2}}{\sqrt{2} - 2}$$

Multiply both sides by (-1):

$$-x = \frac{-(2 - \sqrt{2})}{\sqrt{2} - 2}$$

$$-x = \frac{-2 + \sqrt{2}}{-2 + \sqrt{2}}$$

$$-x = 1$$

$$x = -1$$

PROPERTIES OF STRAIGHT LINES

• PROBLEM 7-24

Find the slope of $f(x) = 3x + 4$.

Solution: Two points on the line determined by $f(x) = 3x + 4$ are $A(0,4)$ and $B(1,7)$.

$$\frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{7-4}{1-0} = 3$$

Note that the ordinates are the y-coordinates and the abscissae are the x-coordinates. The slope determined by points A and B is 3. Hence, the slope of $f(x) = 3x + 4$ is 3. In general, the slope of a linear function of the form $f(x) = mx + b$ is m.

• PROBLEM 7-25

Find the slope and Y-intercept of the following lines.

(a) $y = 3x - 1$ (b) $y = 1 - 4x$ (c) $2y = 4x + 7$

Solution: a) The equation of a line is: $y = mx + b$, where m is the slope of the line and b is the y-intercept of the line. Hence, the line $y = 3x - 1$ has slope = 3 and y-intercept = -1.

b) The line $y = 1 - 4x$ can be rewritten, using the commutative law, as $y = -4x + 1$. Hence, the slope of this line = -4 and the y-intercept = 1.

c) The line $2y = 4x + 7$, after dividing both sides by 2, can be rewritten as:

$$\frac{2y}{2} = \frac{4x + 7}{2}$$

$$y = 2x + \frac{7}{2}$$

Hence, the slope = 2 and the y-intercept = $\frac{7}{2}$.

• PROBLEM 7-26

Find the slope, the y-intercept, and the x-intercept of the equation $2x - 3y - 18 = 0$.

Solution: The equation $2x - 3y - 18 = 0$ can be written in the form of the general linear equation, $ax + by = c$.

$$2x - 3y - 18 = 0$$

$$2x - 3y = 18$$

To find the slope and y-intercept we derive them from the formula of the general linear equation $ax + by = c$. Dividing by b and solving for y we obtain:

$$\frac{a}{b}x + y = \frac{c}{b}$$

$$y = \frac{c}{b} - \frac{a}{b}x$$

where $-\frac{a}{b}$ = slope and $\frac{c}{b}$ = y-intercept.

To find the x-intercept, solve for x and let $y = 0$:

$$x = \frac{c}{b} - \frac{b}{a}y$$

$$x = \frac{c}{a}$$

In this form we have $a = 2$, $b = -3$, and $c = 18$. Thus,

$$\text{slope} = -\frac{a}{b} = -\frac{2}{-3} = \frac{2}{3}$$

$$\text{y-intercept} = \frac{c}{b} = \frac{18}{-3} = -6$$

$$\text{x-intercept} = \frac{c}{a} = \frac{18}{2} = 9$$

• PROBLEM 7-27

The slope and one point of a line are given. Is the Y-intercept positive or negative?

(a) $m = \frac{22}{7}$, $(1,\pi)$ (b) $m = \sqrt{2}$, $(1,1.414)$

Solution: a) The equation of a line is: $y = mx + b$, where m is the slope and b is the y-intercept. Given the slope m and one point of the line, the y-intercept b can be found. Thus it can be determined whether the y-intercept b is positive or negative. For the line with slope $m = 22/7$ and which contains the point $(1,\pi)$:

$$y = mx + b$$

$$\pi = \frac{22}{7}(1) + b$$

$$\pi = \frac{22}{7} + b \qquad (1)$$

Since π is approximately $\frac{22}{7}$, equation (1) becomes:

$$\frac{22}{7} = \frac{22}{7} + b$$

Subtract 22/7 from both sides to obtain:

$$\frac{22}{7} - \frac{22}{7} = \cancel{\frac{22}{7}} + b - \cancel{\frac{22}{7}}$$

$$0 = b$$

Hence, the y-intercept b is neither positive nor negative, since the y-intercept $b = 0$.

b) For the line with slope $m = \sqrt{2}$ and which contains the point $(1,1.414)$:

$$y = mx + b$$

$$1.414 = \sqrt{2}(1) + b$$

$$1.414 = \sqrt{2} + b \qquad (2)$$

Since $\sqrt{2}$ is approximately 1.414, equation (2) becomes:

$$1.414 = 1.414 + b$$

Subtract 1.414 from both sides to obtain:

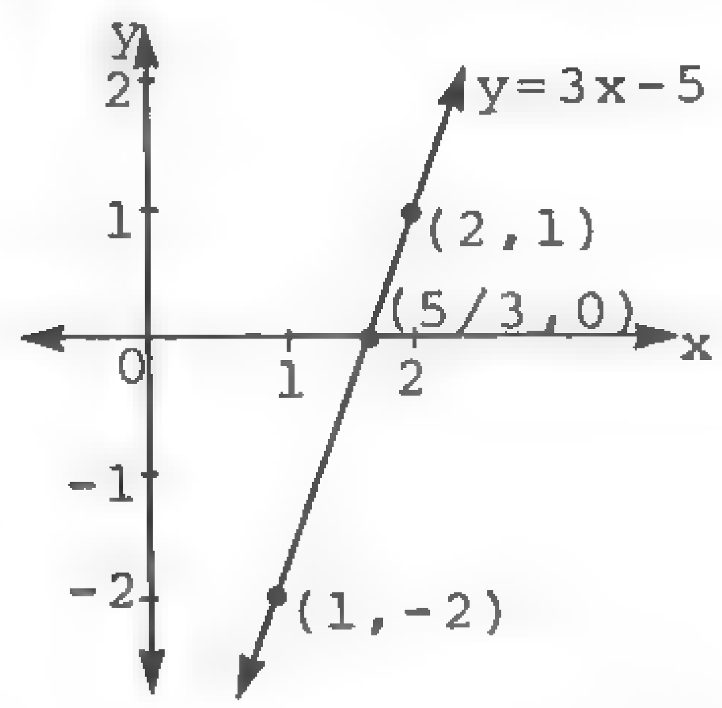
$$1.414 - 1.414 = \cancel{1.414} + b - \cancel{1.414}$$

$$0 = b$$

Again, the y-intercept b is neither positive nor negative, since $b = 0$.

• PROBLEM 7-28

a) Find the zeros of the function f if $f(x) = 3x - 5$.
b) Sketch the graph of the equation $y = 3x - 5$.



Solution: a) The zeros of a function are the numbers for which the value of the function is 0. Therefore, let $f(x) = 3x - 5 = 0$. Solving this equation:

$$3x - 5 = 0$$

Add 5 to both sides of this equation.

$$3x - 5 + 5 = 0 + 5$$
$$3x = 5$$

Divide both sides of this equation by 3.

$$\frac{3x}{3} = \frac{5}{3}$$

$$x = \frac{5}{3}$$

This number is the only zero of f (see the graph of f in the figure).

b) Note that the equation of a line is: $y = mx + b$ where m is the slope of the line and b is the y-intercept. Since the given equation is in this form, the graph is a line. It is only necessary to find two points of the graph in order to draw it. Let $x = 1$. Then $f(x) = f(1) = 3(1) - 5 = 3 - 5 = -2$. Hence, one point is $(1,-2)$. Let $x = 2$. Then $f(x) = f(2) = 3(2) - 5 = 6 - 5 = 1$. Therefore, $(2,1)$ is the other point. These two points determine the straight line shown in the figure.

LINEAR INEQUALITIES

• PROBLEM 7-29

Solve $7\left(\frac{2}{3}x - 1\right) > 2(x - 6)$

Solution: $7\left(\frac{2}{3}x - 1\right) > 2(x - 6)$ (1)

$\frac{14}{3}x - 7 > 2x - 12$ (2) Distributive Property

Subtract $2x$ from both sides of inequality (2).

$$\frac{14}{3}x - 7 - 2x > 2x - 12 - 2x$$

Therefore: $\frac{8}{3}x - 7 > -12$ (3).

Add 7 to both sides of inequality (3).

$$\frac{8}{3}x - 7 + 7 > -12 + 7$$

Therefore: $\frac{8}{3}x > -5$ (4).

Multiply both sides of inequality (4) by $\frac{3}{8}$.

Multiplying both sides of an inequality by a positive number does not change the direction of the inequality. Therefore:

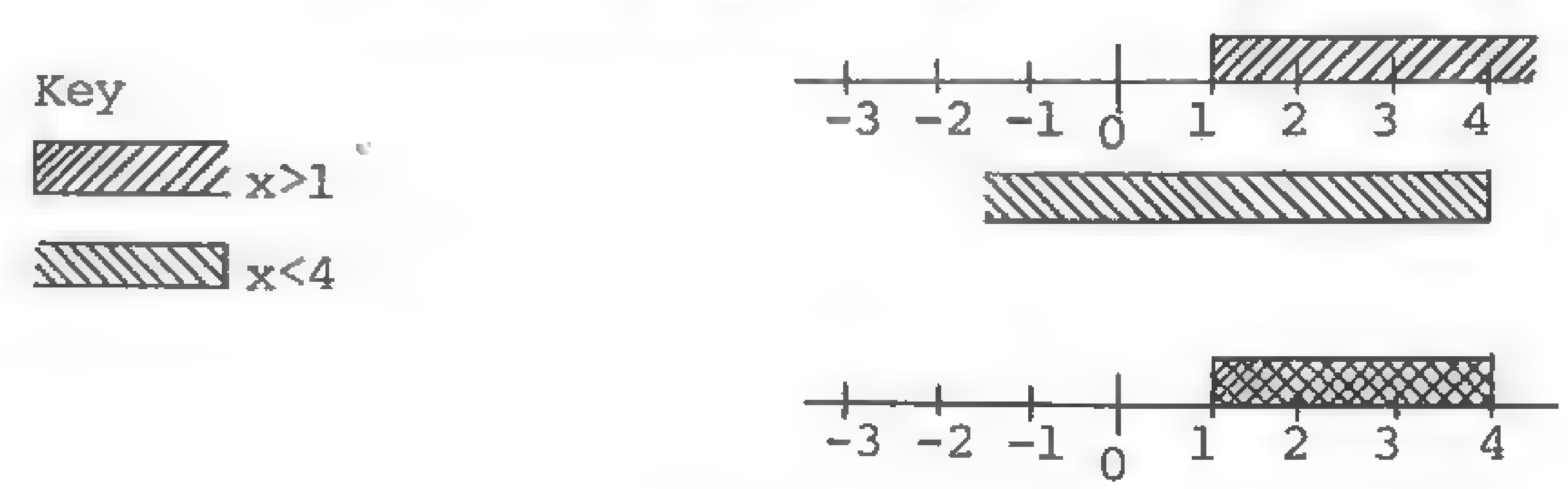
$$\frac{3}{8}\left(\frac{8}{3}x\right) > \frac{3}{8}(-5) \quad \text{and} \quad x > -\frac{15}{8}.$$

Hence, x is any real number greater than $-\frac{15}{8}$.

• PROBLEM 7-30

Solve the inequality

$$\frac{1}{x-1} > \frac{1}{3}.$$



Solution: Since the fraction $\frac{1}{x - 1} > \frac{1}{3}$; that is, since the fraction $\frac{1}{x - 1}$ is greater than 0, $x - 1$ must be positive. Hence, $x - 1 > 0$. If both sides of the given equation are multiplied by $3(x - 1)$, then:

$$3(x - 1)\frac{1}{x - 1} > 3(x - 1)\left(\frac{1}{3}\right)$$

$$3 > x - 1 .$$

Note that multiplying both sides of an inequality by a positive number (in this case, $3(x - 1)$) does not change the sign of the inequality.

Now we have the double restrictions

$$x - 1 > 0 \quad \text{and} \quad 3 > x - 1$$

and the solution set is the intersection of the solution sets of these two inequalities. Solving each of the two inequalities, we find that

$$x > 1 \quad \text{and} \quad x < 4$$

The solution set is the intersection of the two inequalities, as can be seen on a number line (see diagrams). The intersection of these two inequalities is the set $1 < x < 4$. Hence, the solution set is:

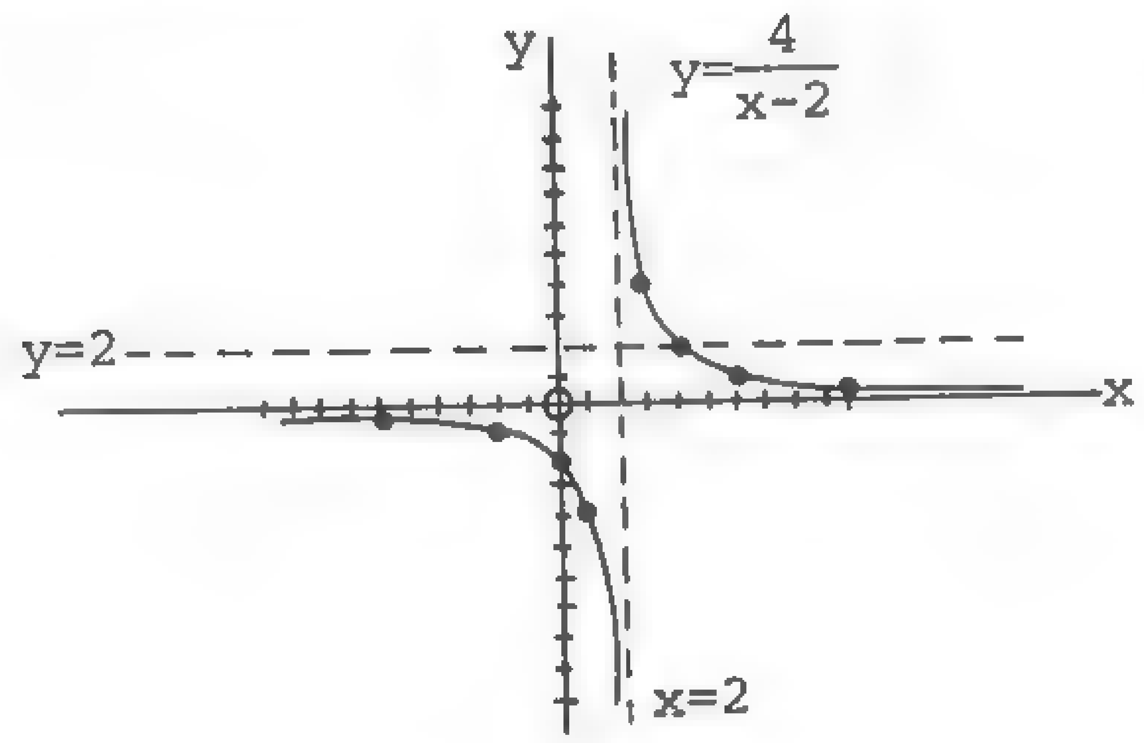
$$X = \{x | 1 < x < 4\}$$

(The endpoints $X = 1$ and $X = 4$ are not included in the solution set).

• PROBLEM 7-31

Solve the inequality

$$\frac{4}{x - 2} < 2.$$



Solution: The inequality is meaningless for $x = 2$ because when $x = 2$ the denominator of the left member is 0, making the fraction undefined.

If $x > 2$, $x - 2$ is positive (since $x > 2$ is equivalent to $x - 2 > 0$), and multiplication of the given inequality by $x - 2$ yields

$$4 < 2(x - 2)$$
$$4 < 2x - 4$$
$$8 < 2x$$
$$4 < x$$
$$x > 4.$$

Thus, the solution is the intersection of $x > 2$ and $x > 4$, $x > 2 \cap x > 4$, which is $\{x | x > 4\}$.

If $x < 2$, $x - 2$ is negative (since $x < 2$ is equivalent to $x - 2 < 0$), and multiplication by $x - 2$ yields

$$4 > 2(x - 2)$$

because multiplication by a negative number reverses an inequality.

Distributing, $\quad 4 > 2x - 4$

Adding 4 to both sides,

$$8 > 2x$$

Dividing both sides by 2,

$$4 > x, \quad \text{or} \quad x < 4.$$

Thus the solution is the intersection of $x < 2$ and $x < 4$, $x < 2 \cap x < 4$, which is $\{x | x < 2\}$. Hence

$$\frac{4}{x - 2} < 2$$

if $x < 2$ or if $x > 4$.

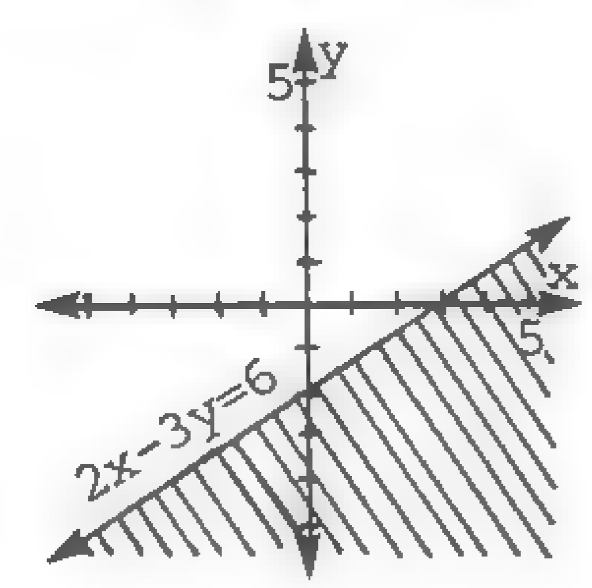
A graphical solution of the problem (see diagram) can be obtained by sketching the equilateral hyperbola $y = 4/(x - 2)$ and the line $y = 2$. The hyperbola may be sketched from its vertical asymptote $x = 2$, its horizontal asymptote $y = 0$, its intercepts $x = 0$, $y = -2$, and a few other points obtained by substitution and symmetry. It is then possible to observe the values of x for which the hyperbola is below the line, namely,

$$x < 2 \quad \text{and} \quad x > 4.$$

The same diagram also shows that $[4/(x - 2)] > 2$ for $2 < x < 4$.

• PROBLEM 7-32

Solve $2x - 3y \geq 6$



Solution: The statement $2x - 3y \geq 6$ means $2x - 3y$ is greater than or equal to 6. Symbolically, we have $2x - 3y > 6$ or $2x - 3y = 6$. Consider the corresponding equality and graph $2x - 3y = 6$. To find the x-intercept,

set y = 0

$$2x - 3y = 6$$
$$2x - 3(0) = 6$$
$$2x = 6$$
$$x = 3$$

{3,0} is the x-intercept.

To find the y-intercept, set x=0

$$2x - 3y = 6$$
$$2(0) - 3y = 6$$
$$-3y = 6$$
$$y = -2$$

{0,-2} is the y-intercept.

A line is determined by two points. Therefore draw a straight line through the two intercepts {3,0} and {0,-2}. Since the inequality is mixed, a solid line is drawn through the intercepts. This line represents the part of the statement $2x - 3y = 6$.

We must now determine the region for which the inequality $2x - 3y > 6$ holds.

Choose two points to decide on which side of the line the region $x - 3y > 6$ lies. We shall try the points (0,0) and (5,1).

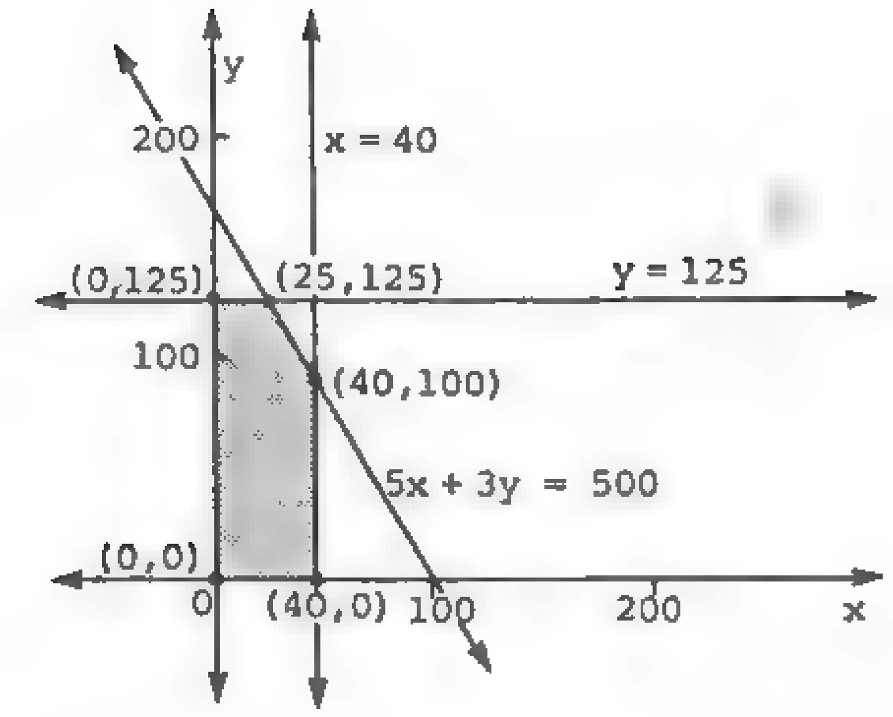
For (0,0)	For (5,1)
$2x - 3y > 6$	$2x - 3y > 6$
$2(0) - 3(0) > 6$	$2(5) - 3(1) > 6$
$0 - 0 > 6$	$10 - 3 > 6$
$0 > 6$	$7 > 6$
False	True

The inequality, $2x - 3y > 6$, holds true for the point (5,1). We shade this region of the xy-plane. That is, the area lying below the line $2x - 3y = 6$ and containing (5,1).

Therefore, the solution contains the solid line, $2x - 3y = 6$, and the part of the plane below this line for which the statement $2x - 3y > 6$ holds.

• PROBLEM 7-33

A livestock farmer has 500 acres to devote to grazing. He estimates that cattle require 5 acres per head and sheep require 3 acres per head. He has winter shelter facilities for 40 head of cattle and for 125 sheep. What constraints are imposed on the number of cattle and sheep he can raise?



Solution: Let x represent the number of cattle raised and y the number of sheep. Since he cannot raise a negative number of either cattle or sheep, we have the constraints

$$x \geq 0 \qquad (1)$$
$$y \geq 0 \qquad (2)$$

Since 5x acres are required for the cattle and 3y acres for the sheep and there are only 500 acres available, we have

$$5x + 3y \leq 500 \qquad (3)$$

Since he can winter only 40 cattle,

$$x \leq 40 \qquad (4)$$

Since he can winter only 125 sheep,

$$y \leq 125 \qquad (5)$$

Relations (1) through (5) are the constraints.

The graph of the constraints in this example is a convex set of points. The corner points of the shaded polygon are (0,0), (40,0), (40,100), (25,125), and (0,125).

• PROBLEM 7-34

Solve the inequality $|5 - 2x| > 3$.

Solution: The property of absolute values states that $|a| = +a$ or $|a| = -a$. Therefore: $|5 - 2x| = 5 - 2x$ or $-(5 - 2x)$. Thus, the given inequality becomes two new inequalities:

$$5 - 2x > 3, \quad -(5 - 2x) > 3.$$

Now, we must solve for x in both inequalities. For the first, we subtract 5 from both sides of the inequality, and then divide by -2. We must keep in mind that division or multiplication by a negative number reverses the inequality sign. Thus, for $5 - 2x > 3$ we have:

$$5 - 5 - 2x > 3 - 5$$
$$-2x > -2$$

$$\frac{-2x}{-2} > \frac{-2}{-2}$$

$$x < 1.$$

For the second inequality, we first take the negative of all the terms inside the parentheses. Thus, for $-(5 - 2x) > 3$ we have:

$$-5 + 2x > 3.$$

Now, we add 5 to both sides of the inequality, and then divide by 2. Thus, we obtain:

$$-5 + 5 + 2x > 3 + 5$$

$$2x > 8$$

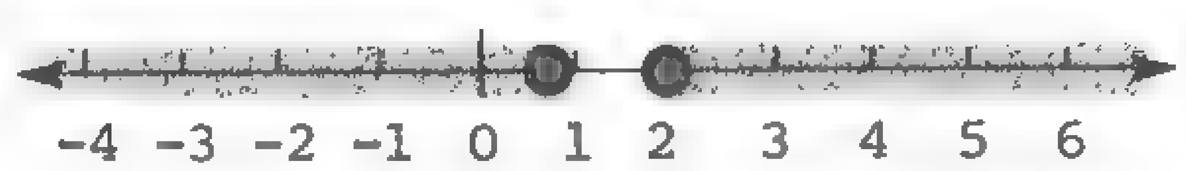
$$\frac{2x}{2} > \frac{8}{2}$$

$$x > 4.$$

Therefore, the above inequality holds when $x < 1$, and when $x > 4$.

• PROBLEM 7-35

Graph $\{x: |3x - 4| \ge 2\}$.



Solution: In general, the required graph of $\{x: |ax + b| \ge c\}$ is the union of two sets: $\{x: ax + b \ge c\} \cup \{x: ax + b \le -c\}$. Therefore, the required graph of $\{x: |3x - 4| \ge 2\}$ is the union of two sets:

$$\{x: 3x - 4 \ge 2\} \cup \{x: 3x - 4 \le -2\}$$

$$3x - 4 \ge 2 \quad \text{or} \quad 3x - 4 \le -2$$

$$3x \ge 6 \quad \text{or} \quad 3x \le 2$$

$$x \ge 2 \qquad x \le 2/3$$

The solution set of $\{x: |3x - 4| \ge 2\}$ is

$$\{x: x \ge 2\} \cup \{x: x \le 2/3\}$$

The graph of the solution set is the union of two rays. Notice that the shaded circles above 2/3 and 2 on the number line indicate that 2/3 and 2 are included in the solution set.

SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

• PROBLEM 7-36

Solve the simultaneous equations $2x + 4y = 11$, $-5x + 3y = 5$ by the method of substitution and by the method of elimination by addition.

Solution: The method of substitution involves solving for one variable in terms of the other and then substituting the obtained value into the second equation. Thus, we solve the first equation for x and substitute in the second:

$$2x + 4y = 11$$
$$2x = 11 - 4y$$
$$x = \frac{11 - 4y}{2}$$

Replacing x by $\left(\frac{11 - 4y}{2}\right)$ in the second equation,

$$-5\left(\frac{11 - 4y}{2}\right) + 3y = 5$$

$$\frac{-55 + 20y}{2} + 3y = 5$$

$$\frac{-55}{2} + 10y + 3y = 5$$

Multiply both sides by 2,

$$-55 + 20y + 6y = 10$$
$$26y = 65$$
$$y = \frac{65}{26} = \frac{5}{2}.$$

Substituting this value for y into the first equation:

$$2x + 4\left(\frac{5}{2}\right) = 11$$

$$2x + 10 = 11$$
$$2x = 1$$
$$x = \frac{1}{2}.$$

We obtain the same result by the method of elimination by addition.

$$2x + 4y = 11 \qquad (1)$$
$$-5x + 3y = 5 \qquad (2)$$

Multiplying equation (1) by 5 and equation (2) by 2 and adding the result we obtain:

$$10x + 20y = 55$$
$$-10x + 6y = 10$$
$$26y = 65$$
$$y = \frac{65}{26} = \frac{5}{2}$$

Once again, replacing y by $\frac{5}{2}$ in equation (1):

$$2x + 4\left(\frac{5}{2}\right) = 11$$

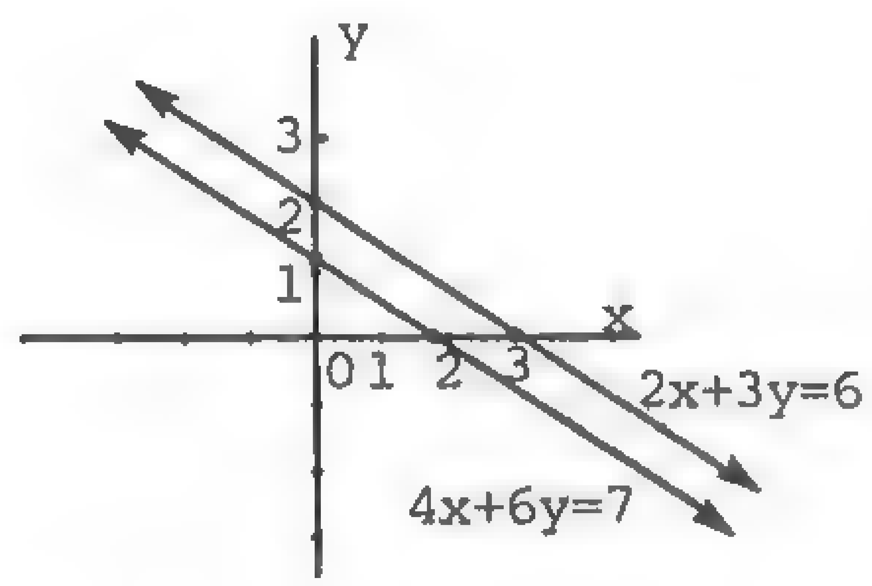
$$2x + 10 = 11$$

$$2x = 1$$
$$x = \frac{1}{2}$$

Thus $\left\{\left(\frac{1}{2}, \frac{5}{2}\right)\right\}$ is the solution to the given system of equations.

• PROBLEM 7-37

Solve the equations $2x + 3y = 6$ and $4x + 6y = 7$ simultaneously.



Solution: We have 2 equations in 2 unknowns,

$$2x + 3y = 6 \qquad (1)$$

and

$$4x + 6y = 7 \qquad (2)$$

There are several methods to solve this problem. We have chosen to multiply each equation by a different number so that when the two equations are added, one of the variables drops out. Thus

multiplying equation (1) by 2: $\quad 4x + 6y = 12 \qquad (3)$

multiplying equation (2) by -1: $\quad \underline{-4x - 6y = -7} \qquad (4)$

adding equations (3) and (4): $\quad 0 = 5$

We obtain a peculiar result!

Actually, what we have shown in this case is that if there were a simultaneous solution to the given equations, then 0 would equal 5. But the conclusion is impossible; therefore there can be no simultaneous solution to these two equations, hence no point satisfying both.

The straight lines which are the graphs of these equations must be parallel if they never intersect, but not identical, which can be seen from the graph of these equations (see the accompanying diagram).

• PROBLEM 7-38

Solve the equations $2x + 3y = 6$ and $y = -(2x/3) + 2$ simultaneously.

Solution: We have 2 equations in 2 unknowns,

$$2x + 3y = 6 \qquad (1)$$

$$y = -(2x/3) + 2 \qquad (2)$$

There are several methods of solution for this problem. Since equation (2) already gives us an expression for y, we use the method of substitution. Substituting $-(2x/3) + 2$ for y in the first equation:

$$2x + 3\left(-\frac{2x}{3} + 2\right) = 6$$

Distributing, $\quad 2x - 2x + 6 = 6$

$$6 = 6$$

Apparently we have gotten nowhere! The result $6 = 6$ is true, but indicates no solution. Actually, our work shows that no matter what real number x is, if y is determined by the second equation, then the first equation will always be satisfied.

The reason for this peculiarity may be seen if we take a closer look at the equation $y = -(2x/3) + 2$. It is equivalent to $3y = -2x + 6$, or $2x + 3y = 6$.

In other words, the two equations are equivalent. Any pair of values of x and y which satisfies one satisfies the other.

It is hardly necessary to verify that in this case the graphs of the given equations are identical lines, and that there are an infinite number of simultaneous solutions of these equations.

• PROBLEM 7-39

Solve the system of equations,

$$2x - y - 4z = 3 \qquad (1)$$
$$-x + 3y + z = 10 \qquad (2)$$
$$3x + 2y - 2z = 2 \qquad (3)$$

Solution: To solve a system of 3 equations in 3 unknowns, we first reduce it to a system of 2 equations in 2 unknowns, a process which can often be done many ways. Although various other algebraic manipulations may be used to arrive at the same result, we will employ the following method: Multiplying equation 1 by (-1) we obtain,

$$2x + y + 4z = 3 \qquad (4)$$

Adding equations (4), (2), and (3) we obtain,

$$-2x + y + 4z = -3$$
$$-x + 3y + z = 10$$
$$\underline{3x + 2y - 2z = 2}$$
$$6y + 3z = -15 \qquad (5)$$

Multiplying equation (2) by 3 we obtain,

$$-3x + 9y + 3z = -30 \qquad (6)$$

Adding equations (6) and (3) we obtain,

$$-3x + 9y + 3z = -30$$
$$\underline{3x + 2y - 2z = -2}$$
$$11y + z = -32 \qquad (7)$$

Multiplying equation (7) by (-3) we obtain,

$$-33y - 3z = 96 \qquad (8)$$

Adding equations (8) and (5) we obtain,

$$\begin{array}{r} -33y - 3z = 96 \\ \underline{6y + 3z = -15} \\ -27y = 81 \\ y = -3 \end{array}$$

Solving for z, we replace, y by (-3) in equation (5):

$$6y + 3z = -15$$
$$6(-3) + 3z = -15$$
$$-18 + 3z = -15$$
$$3z = 3$$
$$z = 1$$

Solving for x, we replace y by (-3) and z by 1 in equation (1):

$$2x - y - 4z = 3$$
$$2x - (-3) - 4(1) = 3$$
$$2x + 3 - 4 = 3$$
$$2x - 1 = 3$$
$$2x = 4$$
$$x = 2$$

Thus the solution to this system is x = 2, y = -3, and z = 1.

Check: Replace x,y, and z by 2, -3, and 1 in each equation.

$$2x - y - 4z = 3 \qquad (1)$$
$$2(2) - (-3) - 4(1) = 3$$
$$4 + 3 - 4 = 3$$
$$3 = 3$$

$$-x + 3y + z = -10 \qquad (2)$$
$$-(2) + 3(-3) + 1 = -10$$
$$-2 - 9 + 1 = -10$$

$$-10 = -10$$

$$3x + 2y - 2z = -2 \qquad (3)$$
$$3(2) + 2(-3) -2(1)= -2$$
$$6 - 6 - 2 = -2$$
$$-2 = -2$$

• PROBLEM 7-40

Solve the system

$$2x - y + 4z = 1 \qquad (1)$$
$$x - y + z = 0 \qquad (2)$$
$$x + y + z = 1 \qquad (3)$$

Solution: It is easiest to eliminate the variable y since the equations (1), (2), (3) differ only by a factor of +1 or - 1 for the variable y. (For the other variables x and z, the equations differ by factors of $\pm$ 2 for x and $\pm$ 4 for z).

Multiplying equation (1) by - 1 we obtain:

$$-2x + y - 4z = -1 \qquad (4)$$
$$x - y + z = 0 \qquad (2)$$
$$x + y + z = 1 \qquad (3)$$

Add equations (4) and (2) to eliminate the variable y and we obtain a new equation (5) in x and z.

$$\begin{array}{rr} -2x + y - 4z = -1 & (4) \\ \underline{x - y + z = 0} & (2) \\ -x \quad - 3z = -1 & (5) \end{array}$$

Add (2) and (3) to obtain another equation (6) in the variables x and z.

$$\begin{array}{rr} x - y + z = 0 & (2) \\ \underline{x + y + z = 1} & (3) \\ 2x \quad + 2z = 1 & (6) \end{array}$$

Now we have a new system of 2 equations in 2 unknowns x and z:

$$-x - 3z = -1 \qquad (5)$$
$$2x + 2z = 1 \qquad (6)$$

We must solve for one variable. The simplest way is to eliminate x. Multiply equation (5) by 2 and we obtain:

$$-2x - 6z = -2 \qquad (7)$$

$$2x + 2z = 1 \qquad (6)$$

Add equations (7) and (6) to obtain:

$$-4z = -1 \qquad (8)$$

Divide equation (8) by -4 to solve for z.

$$z = \frac{1}{4}$$

Substitute z into either (5) or (6) to find x. For equation (5) then we have:

$$-x - 3z = -1 \qquad (5)$$

$$z = \frac{1}{4}$$

$$-x - 3\left(\frac{1}{4}\right) = -1$$

$$-x - \frac{3}{4} = -1$$

$$x = \frac{1}{4}$$

Given x and z we can now solve for y by substituting x and z into any of the three original equations (1), (2) or (3). For equation (2)

$$x - y + z = 0 \qquad (2)$$

$$x = \frac{1}{4}$$

$$z = \frac{1}{4}$$

$$\frac{1}{4} - y + \frac{1}{4} = 0$$

$$-y + \frac{2}{4} = 0$$

$$y = \frac{1}{2}$$

The solution of the original system is then:
$x = \frac{1}{4}$; $y = \frac{1}{2}$; $z = \frac{1}{4}$.

To check, substitute the solution into each of the three original equations (1), (2), and (3).

For (1): $$2x - y + 4z = 1 \qquad (1)$$

$$2\left(\frac{1}{4}\right) - \frac{1}{2} + 4\left(\frac{1}{4}\right) = 1$$

$$\frac{1}{2} - \frac{1}{2} + 1 = 1$$

$$1 = 1$$

For (2): $$x - y + z = 0 \qquad (2)$$

$$\frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

$$\frac{2}{4} - \frac{1}{2} = 0$$

$$\frac{2}{4} - \frac{2}{4} = 0$$

$$0 = 0$$

For (3): $$x + y + z = 1 \qquad (3)$$

$$\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

$$\frac{2}{4} + \frac{2}{4} = 1$$

$$1 = 1$$

• PROBLEM 7-41

Solve for x, y and z:

$$5x + y - z = 9\,, \qquad (1)$$

$$3x + y + 2z = 17\,, \qquad (2)$$

$$x + 2y + 3z = 20. \qquad (3)$$

Solution: Subtract (2) from (1):

$$\begin{array}{r} 5x + y - z = 9 \\ -\ (3x + y + 2z = 17) \\ \hline 2x - 3z = -8 \end{array} \qquad (4)$$

Multiply (2) by 2: $6x + 2y + 4z = 34.$ (5)

Subtract (3) from (5):

$$\begin{array}{r} 6x + 2y + 4z = 34 \\ -\ (x + 2y + 3z = 20) \\ \hline 5x + z = 14 \end{array} \qquad (6)$$

Subtract 5x from both sides: $z = 14 - 5x$ (7)

Substitute $(14 - 5x)$ for z in equation (4):

$$2x - 3(14 - 5x) = -8\ .$$

Distribute: $2x - 42 + 15x = -8$

$17x - 42 = -8$

Add 42 to both sides: $17x = 34$

$x = 2$

Substitute 2 for x in equation (7)

$$z = 14 - 5(2) = 14 - 10 = 4 \quad .$$

Therefore, $x = 2$, and $z = 4$.

Substitute in (1): $5(2) + y - 4 = 9$

$10 + y - 4 = 9$

$6 + y = 9$

Subtract 6 from both sides: $y = 3$

Thus, $x = 2$, $y = 3$, $z = 4$.

Check: $5(2) + 3 - 4 = 9$,

$9 = 9$.

$3(2) + 3 + 2(4) = 17$,

$17 = 17$.

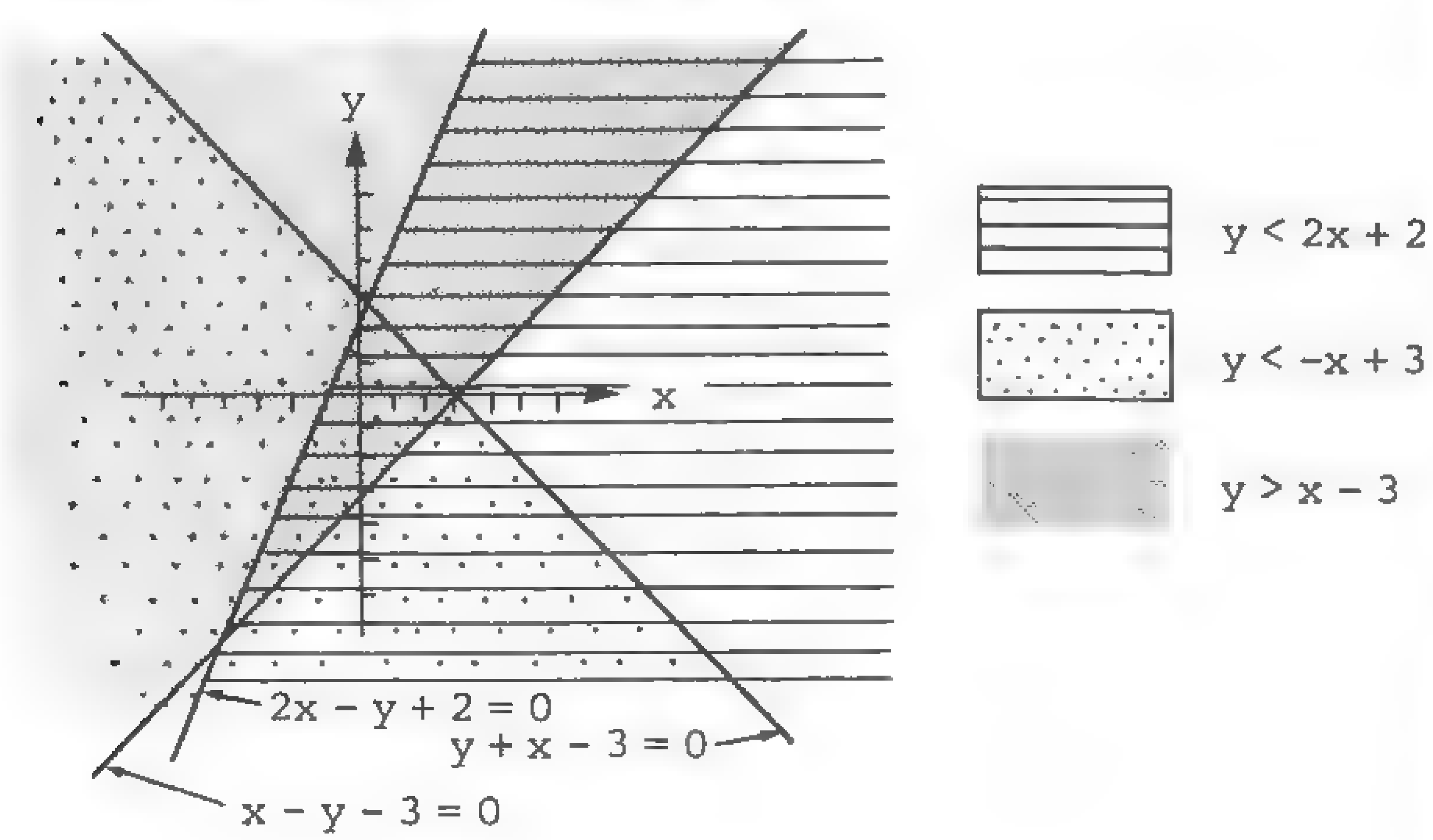
$2 + 2(3) + 3(4) = 20$,

$20 = 20$.

● PROBLEM 7-42

Solve the following system graphically.

$$y - x > -3$$
$$y - 2x < 2$$
$$x + y - 3 < 0$$



Solution: We may rewrite the system:

$$y > x - 3$$
$$y < 2x + 2$$
$$y < -x + 3$$

Graph the linear equation, $y = mx + b$, for each inequality as a straight dotted line. Thus, we graph

$$y = x - 3$$
$$y = 2x + 2$$
$$y = -x + 3$$

To determine in what region of the $x - y$ plane the inequality holds, select points on both sides of the corresponding dotted line and substitute them into the variable statement. Shade in the side of the line whose point makes the inequality a true statement.

The graphs of the variable sentences are represented in the accompanying figure by diagonal, horizontal, and vertical shading, respectively.

The triple-shaded triangular region is the set of all points whose coordinate pairs satisfy all three conditions as defined by the three inequalities in the system.

CHAPTER 8

QUADRATIC EQUATIONS

FACTORING EXPRESSIONS AND FUNCTIONS

• PROBLEM 8-1

Factor A) $4a^2b - 2ab$

B) $9ab^2c^3 - 6a^2c + 12ac$

C) $ac + bc + ad + bd$

Solution: Find the highest common factor of each polynomial.

A) $4a^2b = 2 \cdot 2 \cdot a \cdot a \cdot b$

$2ab = 2 \cdot a \cdot b$

The highest common factor of the two terms is therefore $2ab$. Hence,

$$4a^2b - 2ab = 2ab(2a - 1)$$

B) $9ab^2c^3 = 3 \cdot 3 \cdot a \cdot b \cdot b \cdot c \cdot c \cdot c$

$6a^2c = 3 \cdot 2 \cdot a \cdot a \cdot c$

$12ac = 3 \cdot 2 \cdot 2 \cdot a \cdot c$

The highest common factor of the three terms is $3ac$. Then,

$$9ab^2c^3 - 6a^2c + 12ac = 3ac(3b^2c^2 - 2a + 4)$$

C) An expression may sometimes be factored by grouping terms having a common factor and thus getting new terms containing a common factor. The type form for this case is $ac+bc+ad+bd$, because the terms ac and bc have the common factor c, and ad and bd have the common factor d. Then,

$$ac + bc + ad + bd = c(a + b) + d(a + b)$$

Factoring out $(a + b)$, we obtain:

$$= (a + b)(c + d).$$

• PROBLEM 8-2

Factor $x^2 + 7x + 12$.

Solution: Since $7x = 4x + 3x$, $x^2 + 7x + 12 = x^2 + 4x + 3x + 12$. Factor out the common factor of x from the first two terms. Also, factor out the common factor of 3 from the last two terms. Therefore,

$$x^2 + 7x + 12 = x(x + 4) + 3(x + 4).$$

Now factor out the common factor of $(x + 4)$ from the right side to obtain:

$$x^2 + 7x + 12 = (x + 4)(x + 3).$$

• PROBLEM 8-3

Factor $(z + 1)^2 - b^2$.

Solution: Since $(z + 1)^2 - b^2$ is the difference of two squares we apply the formula for the difference of two squares, $x^2 - y^2 = (x + y)(x - y)$, replacing x by $(z + 1)$ and y by b to obtain:

$$(z + 1)^2 - b^2 = [(z + 1) + b][(z + 1) - b].$$

• PROBLEM 8-4

Factor $x^3 - 8$.

Solution: Since $8 = 2^3$, $x^3 - 8 = (x)^3 - (2)^3$. Therefore, $x^3 - 8$ is the difference of two cubes. The formula for the difference of two cubes is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Replacing a by x and b by 2;

$$\begin{aligned} x^3 - 8 &= (x)^3 - (2)^3 \\ &= (x - 2)\left[x^2 + (x)(2) + (2)^2\right] \\ x^3 - 8 &= (x - 2)(x^2 + 2x + 4) \end{aligned}$$

Simplify $\frac{a^2 - 3ab + 2b^2}{2b^2 + ab - a^2}$.

Solution: Factoring the numerator and denominator of the given fraction we obtain:

$$\frac{a^2 - 3ab + 2b^2}{2b^2 + ab - a^2} = \frac{(a - 2b)(a - b)}{(2b - a)(b + a)}$$

If we negate both factors in the numerator we do not change the fraction's value(because negating both factors gives us -1 multiplied by -1, which equals 1; and multiplication by 1 does not change the expression's value). Thus, we have:

$$\frac{[-(a - 2b)][-(a - b)]}{(2b - a)(b + a)} = \frac{(2b - a)(b - a)}{(2b - a)(b + a)} = \frac{b - a}{b + a},$$

since $\frac{2b - a}{2b - a} = 1$.

SOLVING QUADRATIC EQUATIONS BY FACTORING

• PROBLEM 8-6

Solve: $x^2 - 5x - 14 = 0$.

Solution: To find the roots of this quadratic, we factor it (put it in the form $(x + a)(x + b) = 0$).

Note that $(x + a)(x + b) = x^2 + (a + b)x + ab$

Thus, in our quadratic, $x^2 + (-5)x + (-14)$,

$$a + b = -5 \qquad (1)$$

and

$$ab = -14 \qquad (2)$$

That is, we want the two numbers, a and b, whose sum is (-5), and whose product is (-14).

To find these numbers, we can check the set of numbers whose product is (-14):

(a) $(-14) \times (1) = -14$, therefore equation (2) is satisfied, now check these values in equation (1):

$(-14) + (1) = -13 \neq -5$ therefore we reject these values.

(b) $(-7) \times (2) = -14$, therefore equation (2) is satisfied, now check these values in equation (1):

$(-7) + 2 = -5$ hence both equations are satisfied and we conclude

$$a = -7 \quad \text{and } b = 2.$$

Thus
$$\begin{aligned} x^2 - 5x - 14 &= x^2 + (-7 + 2)x + (-7)(2) \\ &= [x + (-7)][x + 2] \\ &= (x - 7)(x + 2) = 0 \end{aligned}$$

By the fundamental principle, if the product of two numbers $yz = 0$, then either $y = 0$ or $z = 0$; hence if

$$(x - 7)(x + 2) = 0$$

either $x - 7 = 0$ or $x + 2 = 0$

add 7 to both sides | subtract 2 from both sides

$x = 7$ or $x = -2$

This proves that if the equation has roots, they must be either 7 or − 2. We check these values by substituting in the given equation:

If $x = 7$, then
$$\begin{aligned} x^2 - 5x - 14 &= (7)^2 - 5(7) - 14 \\ &= 49 - 35 - 14 \\ &= 49 - 49 \\ &= 0 \end{aligned}$$

If $x = -2$, then
$$\begin{aligned} x^2 - 5x - 14 &= (-2)^2 - 5(-2) - 14 \\ &= 4 + 10 - 14 \\ &= 14 - 14 \\ &= 0 \end{aligned}$$

We may now conclude that the solution to our equation is $x = 7$ or $x = -2$.

• PROBLEM 8-7

Find the roots of $x^2 - 3x - 10 = 0$.

Solution: To find the roots of this quadratic, we factor it (put it in the form $(x + a)(x + b) = 0$).

Note that $(x + a)(x + b) = x^2 + (a + b)x + ab$

Thus in our quadratic, $x^2 + (-3)x + (-10)$,

$$a + b = -3 \qquad (1)$$

and $$ab = -10. \qquad (2)$$

That is, we want the two numbers a and b whose sum is (-3), and whose product is (-10).

To find these numbers, we can check the set of numbers whose product is (-10):

(a) $(-10) \times (1) = -10$, therefore equation (2) is satisfied, now check these values in equation (1):
$(-10) + (1) = -9 \neq -3$ therefore we reject these values.

(b) $(-5) \times (2) = -10$, therefore equation (2) is satisfied, now checking these values in equation (1): $(-5) + 2 = -3$.

Hence both equations are satisfied and we conclude

$$a = -5 \qquad \text{and} \qquad b = 2.$$

Thus,
$$\begin{aligned} x^2 - 3x - 10 &= x^2 + (-5 + 2)x + (-5)(2) \\ &= [x + (-5)][x + 2] \\ &= (x - 5)(x + 2) = 0. \end{aligned}$$

Hence, by the fundamental property which states that if $ab = 0$, either $a = 0$ or $b = 0$, $x - 5 = 0$ or $x + 2 = 0$ and

$$x = 5 \qquad \text{or} \qquad x = -2.$$

This proves that if the equation has roots, they must be either 5 or -2. So far we have not proved that these are roots. We can check this by substituting in the given equation. If $x = 5$, then

$$\begin{aligned} x^2 - 3x - 10 &= (5)^2 - 3(5) - 10 \\ &= 25 - 15 - 10 \\ &= 25 - 25 \\ &= 0 \end{aligned}$$

Thus 5 is indeed a root of the equation.

If $x = -2$, then

$$\begin{aligned} x^2 - 3x - 10 &= (-2)^2 - 3(-2) - 10 \\ &= 4 + 6 - 10 \\ &= 10 - 10 \end{aligned}$$

$$= 0$$

Thus -2 is also a root.

Such a check not only has a logical purpose, but it also assures us that we have not made a mistake in arithmetic. We may now conclude that the solution to our equation is $x = 5$ or $x = -2$.

• PROBLEM 8-8

Find the roots of the function G defined by the rule
$$G(x) = x^2 + 5x + 6.$$

Solution: To find the roots of the function G, find those values of x which satisfy $G(x) = 0$. Let $x^2 + 5x + 6 = 0$. In factored form this may be written $(x + 3)(x + 2) = 0$. Values of x which make this product = 0 satisfy $x + 3 = 0$ or $x + 2 = 0$. Hence $x = -3$ or $x = 2$.

Check:

for $x = -3$: $(-3)^2 + 5(-3) + 6 = 9 - 15 + 6 = 0$

for $x = -2$: $(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$

• PROBLEM 8-9

Solve the equation $6x^2 = 2 - x$.

Solution: Write the equation in standard quadratic form by adding $x - 2$ to both sides of the equation. Then we have $6x^2 + x - 2 = 0$. In factored form this becomes $(3x + 2)(2x - 1) = 0$. The values of x that make this product = 0 satisfy

$$\begin{aligned} 3x + 2 &= 0 & &\text{or} & 2x - 1 &= 0 \\ 3x &= -2 & &\text{or} & 2x &= 1 \\ x &= -\tfrac{2}{3} & &\text{or} & x &= \tfrac{1}{2} \end{aligned}$$

Check:

for $x = -\frac{2}{3}$

$$6\left(-\tfrac{2}{3}\right)^2 \stackrel{?}{=} 2 - \left(-\tfrac{2}{3}\right)$$

$$6\left(\frac{4}{9}\right) \stackrel{?}{=} \frac{18}{9} - \left(-\frac{6}{9}\right)$$

$$\frac{24}{9} = \frac{24}{9}$$

for $x = \frac{1}{2}$

$$6\left(\tfrac{1}{2}\right)^2 \stackrel{?}{=} 2 - \left(\tfrac{1}{2}\right)$$

$$6\left(\tfrac{1}{4}\right) \stackrel{?}{=} \frac{8}{4} - \left(\frac{2}{4}\right)$$

$$\frac{6}{4} = \frac{6}{4}$$

Therefore the solution set is $\{-\frac{2}{3}, \frac{1}{2}\}$.

• PROBLEM 8-10

Solve $x^2 - 6x + 8 = 0$.

Solution: This problem may be solved by the method of completing the square: Arrange the equation with the constant term in the right member

$$x^2 - 6x = -8.$$

Take $\frac{1}{2}$ of the coefficient of x, square this, and add the result to both members. Thus, $\frac{1}{2}$ of -6 is -3, and $(-3)^2 = 9$. Add 9 to both members:

$$x^2 - 6x + 9 = -8 + 9 = 1.$$

This procedure makes the left member a perfect square. Factor,

$$(x - 3)^2 = 1.$$

Extract the square root of both members,

$$x - 3 = \pm 1.$$

When $x - 3 = +1$, then $x = 4$ and when $x - 3 = -1$, then $x = 2$.

Check: for $x = 4$: for $x = 2$:

$$4^2 - 6(4) + 8 = 0 \qquad 2^2 - 6(2) + 8 = 0$$

$$16 - 24 + 8 = 0 \qquad 4 - 12 + 8 = 0$$

$$0 = 0 \qquad 0 = 0.$$

Sol: $x = \{4,2\}$.

• PROBLEM 8-11

Solve $3x^2 + 5x - 2 = 0$.

Solution: This problem may be solved by the method of completing the square: First, divide both members by 3, the coefficient of the x^2 term,

$$x^2 + \frac{5}{3}x - \frac{2}{3} = 0.$$

Arrange the equation with the constant term in the right member. Thus,

$$x^2 + \frac{5}{3}x = \frac{2}{3}.$$

Take $\frac{1}{2}$ the coefficient of x, square this, and add the result to both members. Thus, $\frac{1}{2}$ of $\frac{5}{3}$ is $\frac{5}{6}$ and $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$.

Add $\frac{25}{36}$ to both members,

$$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{2}{3} + \frac{25}{36} = \frac{49}{36}.$$

This makes the left member a perfect square. Factor:

$$\left[x + \frac{5}{6}\right]^2 = \frac{49}{36}.$$

Extract the square root of both members

$$x + \frac{5}{6} = \pm \frac{7}{6}.$$

When $x + \frac{5}{6} = \frac{7}{6}$, then $x = \frac{7}{6} - \frac{5}{6} = \frac{2}{6} = \frac{1}{3}$.

When $x + \frac{5}{6} = -\frac{7}{6}$, then $x = -\frac{7}{6} - \frac{5}{6} = -\frac{12}{6} = -2$.

Check: for $x = \frac{1}{3}$:

$$3\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) - 2 = 0$$

$$3\left(\frac{1}{9}\right) + 5\left(\frac{1}{3}\right) - 2 = 0$$

$$\frac{3}{9} + \frac{15}{9} - \frac{18}{9} = 0$$

$$0 = 0.$$

For $x = -2$:

$$3(-2)^2 + 5(-2) - 2 = 0$$

$$3(4) + 5(-2) - 2 = 0$$

$$12 - 10 - 2 = 0$$

$$0 = 0.$$

Sol: $x = \left\{\frac{1}{3}, -2\right\}$.

SOLUTIONS BY QUADRATIC FORMULA

● **PROBLEM 8-12**

Obtain the quadratic equation in standard form that is equivalent to $4x - 3 = 5x^2$.

Solution: The standard form of a quadratic equation is $ax^2 + bx + c = 0$. Starting with our given equation $4x - 3 = 5x^2$, we add $(-5x^2)$ to both members,

$$(4x - 3) + (-5x^2) = 5x^2 + (-5x^2)$$

$$(4x - 3) + (-5x^2) = 0$$

commuting we obtain $-5x^2 + 4x - 3 = 0$

This is the required equation with $a = -5$, $b = 4$, and $c = -3$.

● **PROBLEM 8-13**

Find the roots of the equation $x^2 + 12 - 85 = 0$.

Solution: The roots of this equation may be found using the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

In this equation $A = 1$, $B = 12$, and $C = -85$. Hence, by the quadratic formula,

$$x = \frac{-12 + \sqrt{144 + 340}}{2} \quad \text{or} \quad x = \frac{-12 - \sqrt{144 + 340}}{2}$$

$$x = \frac{-12 + 22}{2} \quad \text{or} \quad x = \frac{-12 - 22}{2}$$

Therefore $x = 5$ or $x = -17$. This is equivalent to the statement that the solution set is $\{-17, 5\}$.

● **PROBLEM 8-14**

Use the quadratic formula to solve for x in the equation $x^2 - 5x + 6 = 0$.

Solution: The quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is used to solve equations in the form $ax^2 + bx + c = 0$. Here $a = 1$, $b = -5$, and $c = 6$. Hence

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm \sqrt{1}}{2}$$

$$= \frac{5 \pm 1}{2}$$

$$= \frac{5 + 1}{2} \quad \text{or} \quad \frac{5 - 1}{2}$$

$$= \frac{6}{2} \quad \text{or} \quad \frac{4}{2}$$

$$= 3 \quad \text{or} \quad 2$$

Thus the roots of the equation $x^2 - 5x + 6 = 0$ are $x = 3$ and $x = 2$.

● **PROBLEM 8-15**

Solve the equation $x^2 + 5x + 6 = 0$ by the quadratic formula.

Solution: We use the quadratic formula, which states

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ for cases}$$

where $ax^2 + bx + c = 0$. For this equation, $a = 1$, $b = 5$, $c = 6$. Therefore the solutions are

$$x = \frac{-5 \pm \sqrt{25 - 4(1)(6)}}{2 \cdot 1}$$

$$= \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm \sqrt{1}}{2}$$

or $$x_1 = \frac{-5 + 1}{2} = -2, \quad x_2 = \frac{-5 - 1}{2} = -3.$$

● **PROBLEM 8-16**

Solve $6x^2 - 7x - 20 = 0$.

Solution: $6x^2 - 7x - 20 = 0$ is not factorable. Therefore, find the roots of the quadratic equation $ax^2 + bx + c$ using:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 6$, $b = -7$, $c = -20$.

$$x = \frac{7 \pm \sqrt{49 - 4(6)(-20)}}{12}$$

$$x = \frac{7 \pm \sqrt{529}}{12} = \frac{7 \pm 23}{12}.$$

Therefore, $x_1 = \frac{7 + 23}{12} = \frac{30}{12} = \frac{5}{2}$

$$x_2 = \frac{7 - 23}{12} = -\frac{16}{12} = -\frac{4}{3}.$$

• PROBLEM 8-17

Solve the equation $2x^2 - 5x + 3 = 0$.

Solution:

(1) $2x^2 - 5x + 3 = 0$

Equation (1) is a quadratic equation of the form $ax^2 + bx + c = 0$ in which $a = 2$, $b = -5$, and $c = 3$.

Therefore, the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

may be used to find the solutions of equation (1). Substituting the values for a, b, and c in the quadratic formula:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{1}}{4}$$

$$x = \frac{5+1}{4} = \frac{3}{2}, \quad \text{and } x = \frac{5-1}{4} = 1$$

Check: Substituting $x = \frac{3}{2}$ in the given equation,

$$2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 = 0$$

$$0 = 0$$

Substituting $x = 1$ in the given equation,

$$2(1)^2 - 5(1) + 3 = 0$$

$$0 = 0$$

• PROBLEM 8-18

Solve $x^2 - 7x + 10 = 0$.

Solution: $x^2 - 7x + 10 = 0$ is a quadratic equation of the form $ax^2 + bx + c = 0$ with $a = 1$, $b = -7$, $c = 10$. The roots of the equation may be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Substituting values for a,b, and c

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 + 4(1)(10)}}{2(1)}.$$

$$x = \frac{7 \pm \sqrt{49 - 4(1)(10)}}{2}$$

$$x = \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$x = \frac{7+3}{2} = 5; \quad x = \frac{7-3}{2} = 2.$$

Check: for $x = 5$, $(5)^2 - 7(5) + 10 = 0$

$$25 - 35 + 10 = 0$$

$$0 = 0$$

for $x = 2$, $(2)^2 - 7(2) + 10 = 0$

$$4 - 14 + 10 = 0$$

$$0 = 0.$$

More simply, the problem could have been solved by factoring:

$$x^2 - 7x + 10 = 0$$

$$(x - 5)(x - 2) = 0.$$

Set each factor equal to zero to find all values of x which make the product = 0.

$x - 5 = 0$	$x - 2 = 0$
$x = 5$	$x = 2$

• **PROBLEM 8-19**

Solve $t^2 - 8t + 3 = 0$ by the quadratic formula.

Solution: Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which applies to the situation where $ax^2 + bx + c = 0$. In our case, $a = 1$, $b = -8$, $c = 3$, and

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - (4 \cdot 1 \cdot 3)}}{2 \cdot 1}$$

$$= \frac{8 \pm \sqrt{64 - 12}}{2 \cdot 1} = \frac{8 \pm \sqrt{52}}{2}$$

$$= \frac{8}{2} \pm \frac{\sqrt{52}}{2} \quad \text{by the definition of addition of fractions}$$

$$= 4 \pm \frac{\sqrt{13 \cdot 4}}{2} \quad \text{because } \frac{8}{2} = 4, \text{ and } 52 = 13 \cdot 4$$

$$= 4 \pm \frac{\sqrt{13} \cdot \sqrt{4}}{2} \quad \text{Recall } \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

$$= 4 \pm \frac{2\sqrt{13}}{2} \quad \text{because } \sqrt{4} = 2$$

$$= 4 \pm \sqrt{13}$$

• **PROBLEM 8-20**

Solve for x: $3x^2 + 5 = 0$.

Solution: This quadratic equation can be solved for x using the quadratic formula, which applies to equations in the form $ax^2 + bx + c = 0$ (in our equation $b = 0$). There is, however, an easier method that we can use: adding -5 to both sides,

$$3x^2 = -5$$

dividing both sides by 3,

$$x^2 = -\frac{5}{3}$$

Taking the square root of both sides,

$$x = \pm\sqrt{-\frac{5}{3}} = \pm\sqrt{(-1)\left(\frac{5}{3}\right)} = \pm\sqrt{-1}\sqrt{\frac{5}{3}}$$

By definition $\sqrt{-1} = i$. Thus,

$$\pm\sqrt{-1}\sqrt{\frac{5}{3}} = \pm i\sqrt{\frac{5}{3}} = \pm i\sqrt{\frac{5}{3} \cdot \frac{3}{3}} = \pm i\sqrt{\frac{15}{9}} = \pm i\frac{\sqrt{15}}{\sqrt{9}} = \pm\frac{i\sqrt{15}}{3}.$$

Thus, $\qquad x = \frac{i\sqrt{15}}{3}, \; -\frac{i\sqrt{15}}{3}.$

• **PROBLEM 8-21**

Use the quadratic formula to solve $2x^2 - 5x + 8 = 0$.

Solution: Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which applies to the situation where $ax^2 + bx + c = 0$. In our case $a = 2$, $b = -5$, and $c = 8$. Substituting these values into the quadratic formula we obtain:

$$x = \frac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot 8}}{2 \cdot 2}$$

$$= \frac{5 \pm \sqrt{25 - 64}}{4} = \frac{5 \pm \sqrt{-39}}{4}$$

Note that the $\sqrt{-39}$, the square root of a negative number, is not defined for real numbers, hence we must use the imaginary number system.

Since $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, $\sqrt{-39} = \sqrt{(-1) \cdot (39)} = \sqrt{-1} \cdot \sqrt{39}$

By definition $i = \sqrt{-1}$, so $\sqrt{-39} = i\sqrt{39}$

Therefore $\qquad x = \frac{5 \pm i\sqrt{39}}{4}$

• **PROBLEM 8-22**

Solve $x^2 + 2x + 5 = 0$.

Solution: $x^2 + 2x + 5 = 0$ is a nonfactorable quadratic equation of the form $ax^2 + bx + c = 0$. Therefore, to find the roots of the equation use the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ with } a = 1, b = 2, c = 5.$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

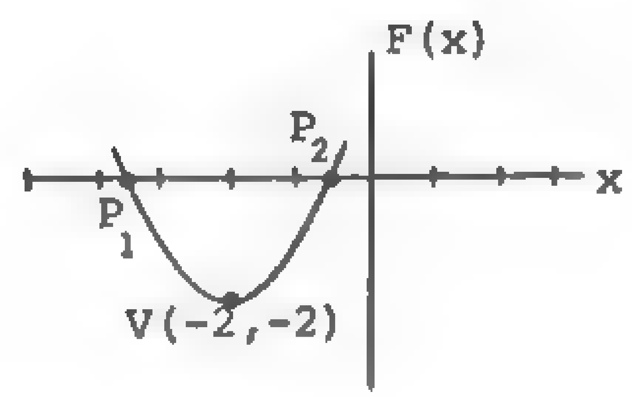
$$x = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{-1} \cdot \sqrt{16}}{2}.$$

In this case the roots involve imaginary numbers. The result can be simplified by using $i = \sqrt{-1}$ to give

$$x = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

• PROBLEM 8-23

> Find the roots of the function F whose rule of correspondence is $F(x) = 2x^2 + 8x + 4$.



Solution: The roots of the function F are those values of x which satisfy $F(x) = 0$. Therefore we seek the solution set of $2x^2 + 8x + 4 = 0$, or $x^2 + 4x + 2 = 0$ (dividing both sides of the equation by 2). Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$, $b = 4$, and $c = 2$, we have:

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 8}}{2}$$

$$x = \frac{-4 \pm \sqrt{4 \cdot 2}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

Hence the roots are $x_1 = -2 + \sqrt{2}$ and $x_2 = -2 - \sqrt{2}$.

The graph of the quadratic function F is a parabola and intersects the domain axis at the points $P_1(-2-\sqrt{2},0)$ and $P_2(-2+\sqrt{2},0)$.

• PROBLEM 8-24

> Find the sum and product of the roots of the equation
>
> $$3x^2 - 2x + 1 = 0.$$

Solution: The given equation is a quadratic equation in which $a = 3$, $b = -2$, and $c = 1$. Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the given equation:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)} = \frac{2 \pm \sqrt{-8}}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{4}\sqrt{-2}}{6} = \frac{1}{3} \pm \frac{2\sqrt{-2}}{6}$$

$$= \frac{1 \pm \sqrt{-2}}{3}$$

The sum of the roots is:

$$\frac{1 + \sqrt{-2}}{3} + \frac{1 - \sqrt{-2}}{3} = \frac{1 + \sqrt{-2} + 1 - \sqrt{-2}}{3} = \frac{2}{3}.$$

The product of the root is:

$$\left(\frac{1 + \sqrt{-2}}{3}\right)\left(\frac{1 - \sqrt{-2}}{3}\right) = \frac{1 + \sqrt{-2} - \sqrt{-2} - (-2)}{9}$$

$$= \frac{1}{3}.$$

• PROBLEM 8-25

> Determine the quadratic equation whose roots are $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$.

Solution: We can determine the quadratic equation from the sum and the product of the roots. A quadratic equation whose roots are x_1 and x_2 may be written in the form

$$x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = 0$$

where the sum of the roots is $x_1 + x_2 = -\frac{b}{a}$ and the product of the roots is $x_1 \cdot x_2 = \frac{c}{a}$. Here,

$$x_1 = 2 + \sqrt{3} \text{ and } x_2 = 2 - \sqrt{3}.$$

Then, $x_1 + x_2 = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$ and

and $x_1 \cdot x_2 = (2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1.$

Hence, the equation is:

$$x^2 - 4x + 1 = 0.$$

• PROBLEM 8-26

Find the values of the constant k in the equation

$$2x^2 - kx + 3k = 0$$

if the difference of the roots is $\frac{5}{2}$.

Solution: The given equation is a quadratic equation since it is in the form $ax^2 + bx + c = 0$. If both sides of the given equation are divided by 2, then:

$$\frac{2x^2 - kx + 3k}{2} = \frac{0}{2}$$

$$x^2 - \frac{k}{2}x + \frac{3}{2}k = 0 \qquad (1)$$

Equation (1) is in the form $x^2 - Sx + P = 0$, where S = sum of the roots = $\frac{k}{2}$ and P = product of the roots = $\frac{3}{2}k$.

Let the roots be r_1 and r_2. Since the difference of the roots is $\frac{5}{2}$ and the sum of the roots is $\frac{k}{2}$, the following equations result:

$$r_1 - r_2 = \frac{5}{2} \quad (2) \qquad \text{and} \qquad r_1 + r_2 = \frac{k}{2} \qquad (3)$$

Solving for r_1 by adding equations (2) and (3):

$$r_1 - r_2 = \frac{5}{2}$$

$$r_1 + r_2 = \frac{k}{2}$$

$$2r_1 = \frac{5}{2} + \frac{k}{2}$$

$$2r_1 = \frac{5 + k}{2}.$$

Multiplying both sides by $\frac{1}{2}$:

$$\frac{1}{\cancel{2}}\left(\cancel{2}r_1\right) = \frac{1}{2}\left[\frac{5 + k}{2}\right]$$

$$r_1 = \frac{5 + k}{4}$$

$$r_1 = \frac{k + 5}{4} \qquad (4)$$

Solving for r_2 by subtracting equation (3) from equation (2):

$$r_1 - r_2 = \frac{5}{2}$$

$$\left(-\, r_1 + r_2 = \frac{k}{2}\right)$$

$$-2r_2 = \frac{5}{2} - \frac{k}{2}$$

$$-2r_2 = \frac{5 - k}{2} \qquad (5)$$

Multiply both sides of equation (5) by $-\frac{1}{2}$:

$$\left(-\frac{1}{2}\right)\left(-2r_2\right) = \left(-\frac{1}{2}\right)\left(\frac{5 - k}{2}\right)$$

$$r_2 = \frac{(-1)(5 - k)}{(2)(2)}$$

$$= \frac{-5 + k}{4}$$

$$r_2 = \frac{k - 5}{4} \qquad (6)$$

Multiplying equations (4) and (6):

$$r_1r_2 = \left(\frac{k + 5}{4}\right)\left(\frac{k - 5}{4}\right)$$

$$= \frac{k^2 + \cancel{5k} - \cancel{5k} - 25}{16}$$

$$r_1r_2 = \frac{k^2 - 25}{16}.$$

However, it was found earlier that the product, P, of the roots was $\frac{3}{2}k$; that is, $r_1r_2 = \frac{3}{2}k$. Then, setting these two expressions for r_1r_2 equal:

$$\frac{k^2 - 25}{16} = \frac{3}{2}k$$

$$\frac{k^2 - 25}{16} = \frac{3k}{2}.$$

Subtract $\frac{3k}{2}$ from both sides of this equation:

$$\frac{k^2 - 25}{16} - \frac{3k}{2} = \frac{3k}{2} - \frac{3k}{2}$$

$$\frac{k^2 - 25}{16} - \frac{3k}{2} = 0.$$

Obtaining a common denominator of 16 for the two fractions on the left side of the equation:

$$\frac{k^2 - 25}{16} - \frac{8(3k)}{8(2)} = 0$$

$$\frac{k^2 - 25}{16} - \frac{24k}{16} = 0$$

$$\frac{k^2 - 25 - 24k}{16} = 0.$$

$$\frac{k^2 - 24k - 25}{16} = 0$$

Multiply both sides of this equation by 16:

$$16\left(\frac{k^2 - 24k - 25}{16}\right) = 16(0)$$

$$k^2 - 24k - 25 = 0.$$

Factor the left side of this equation into a product of two polynomials:

$$(k - 25)(k + 1) = 0 \qquad (7)$$

Whenever a product $ab = 0$, where a and b are any two numbers, either $a = 0$ or $b = 0$. Then, equation (7) becomes:

$$k - 25 = 0 \quad \text{or} \quad k + 1 = 0$$

$$k = 25 \quad \text{or} \quad k = -1.$$

Hence, the two solutions are $k = -1$ and $k = 25$.

Compute the value of the discriminant and then determine the nature of the roots of each of the following four equations:

$$4x^2 - 12x + 9 = 0,$$

$$3x^2 - 7x - 6 = 0$$

$$5x^2 + 2x - 9 = 0$$

and $x^2 + 3x + 5 = 0.$

Solution: The discriminant, the term of the quadratic formula which appears under the radical, is $b^2 - 4ac$. It can be used to determine the nature of the roots of equations in the form $ax^2 + bx + c = 0$. Assuming a,b,c are real numbers, then,

(1) if $b^2 - 4ac > 0$, the roots are real and unequal

(2) if $b^2 - 4ac = 0$, the roots are real and equal

(3) if $b^2 - 4ac < 0$, the roots are imaginary

Assuming a,b,c are real and rational numbers then,

(1) if $b^2 - 4ac$ is a perfect square $\neq 0$, the roots are real, rational and unequal,

(2) if $b^2 - 4ac = 0$, the roots are real, rational, and equal,

(3) if $b^2 - 4ac > 0$, but not a perfect square, the roots are real, irrational and unequal,

(4) if $b^2 - 4ac < 0$, the roots are imaginary.

$$4x^2 - 12x + 9 = 0$$

Here a,b,c are rational numbers,

$$a = 4,\ b = -12 \text{ and } c = 9.$$

Therefore,

$$b^2 - 4ac = (-12)^2 - 4(4)(9) = 144 - 144 = 0$$

Since the discriminant is 0, the roots are rational and equal.

(b) $3x^2 - 7x - 6 = 0$

Here a,b,c are rational numbers,

$a = 3$, $b = -7$, and $c = -6$.

Therefore,

$$b^2 - 4ac = (-7)^2 - 4(3)(-6) = 49 + 72 = 121 = 11^2$$

Since the discriminant is a perfect square, the roots are rational and unequal.

(c) $5x^2 + 2x - 9 = 0$

Here a,b,c are rational numbers,

$a = 5$, $b = 2$ and $c = -9$

Therefore,

$$b^2 - 4ac = 2^2 - 4(5)(-9) = 4 + 180 = 184$$

Since the discriminant is greater than zero, but not a perfect square, the roots are irrational and unequal.

(d) $x^2 + 3x + 5 = 0$

Here a,b,c are rational numbers,
$a = 1$, $b = 3$, and $c = 5$

Therefore,

$$b^2 - 4ac = 3^2 - 4(1)(5) = 9 - 20 = -11$$

Since the discriminant is negative the roots are imaginary.

• PROBLEM 8-28

Discuss the nature of the roots of

(a) $3x^2 - 7x + 3 = 0$ (b) $5x^2 + 3x + 1 = 0$

Solution: Equations (a) and (b) are of the form,

$$ax^2 + bx + c = 0.$$

In (a): $a = 3$, $b = -7$, $c = 3$

In (b): $a = 5$, $b = 3$, $c = 1$.

First we find the value of the discriminant, $b^2 - 4ac$, in each case.

If $b^2 - 4ac > 0$ the roots are real and unequal.

If $b^2 - 4ac = 0$ the roots are real and equal.

If $b^2 - 4ac < 0$ the roots are imaginary.

(a) $b^2 - 4ac = (-7)^2 - (4 \cdot 3 \cdot 3)$
$= 49 - 36$
$= 13.$

Thus, the roots are real and unequal.

(b) $b^2 - 4ac = (3)^2 - (4 \cdot 5 \cdot 1)$
$= 9 - 20$
$= -11.$

Thus, there are no real roots, i.e., the roots are imaginary.

• PROBLEM 8-29

Determine the character of the roots of the equation $4x^2 - 12x + 9 = 0$.

Solution: The given equation is a quadratic equation where $a = 4$, $b = -12$, and $c = 9$. The discriminant of this equation,

$$b^2 - 4ac, \text{ is } 144 - 144 = 0.$$

By the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the only root of the given equation is:

$$x = \frac{-(-12) \pm \sqrt{0}}{2(4)} = \frac{12}{8} = \frac{3}{2}.$$

Since the discriminant is equal to zero and by the root obtained for the given equation, there is only one real rational root.

SOLVING QUADRATIC INEQUALITIES

• PROBLEM 8-30

Solve the inequality $(2x - 1)(x + 2) < 0$.

Solution: Since the two factors must be of opposite sign for their product to be negative, we have the two tentative possibilities:

$2x - 1 < 0, \quad x + 2 > 0,$

or $2x - 1 > 0, \quad x + 2 < 0.$

Solving the first pair of inequalities:

$2x - 1 < 0$ and $x + 2 > 0$

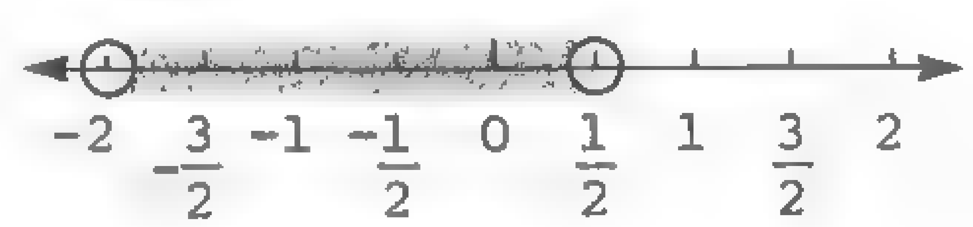
add 1 to both sides: | subtract 2 from both sides:

$2x < 1$ |

divide both sides by 2: |

$x < \frac{1}{2}$ and $x > -2$

Thus, the first pair implies that $x < \frac{1}{2}$ and $x > -2$, or $-2 < x < \frac{1}{2}$; the graph is as follows:



Solving the second pair of inequalities:

$2x - 1 > 0$ and $x + 2 < 0$

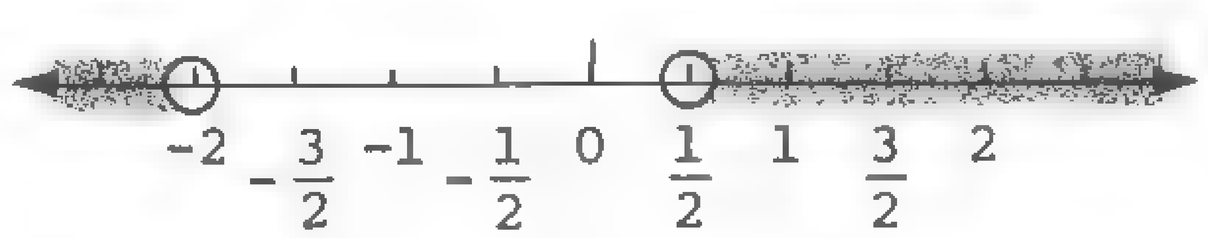
Adding 1 to both sides: |

$2x > 1$ | Subtracting 2 from both sides:

Dividing both sides by 2: |

$x > \frac{1}{2}$ and $x < -2$

Thus, the second pair implies that $x > \frac{1}{2}$ and $x < -2$; the graph is as follows:



• PROBLEM 8-31

Solve the inequality $x^2 - x - 2 \le 0$.

Solution: Factoring the left side of the given inequality,

$$(x - 2)(x + 1) \le 0.$$

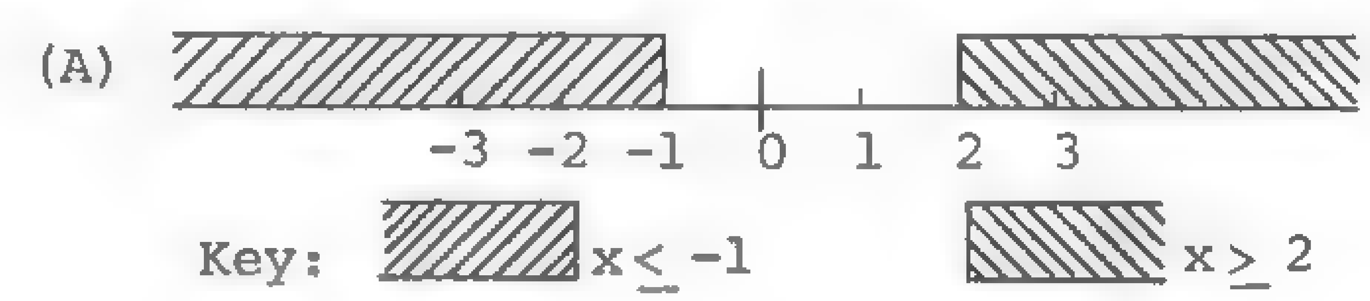
If the product of two numbers is negative, one of the numbers is positive and the other is negative. Hence, there are two cases:

Case 1: $x - 2 \ge 0$, $x + 1 \le 0$

Solving these two inequalities,

$x \ge 2, \quad x \le -1$

Graph these new inequalities on number line (A).



Note that there is no value of x which satisfies both inequalities at the same time since these two inequalities do not intersect anywhere on the number line (A).

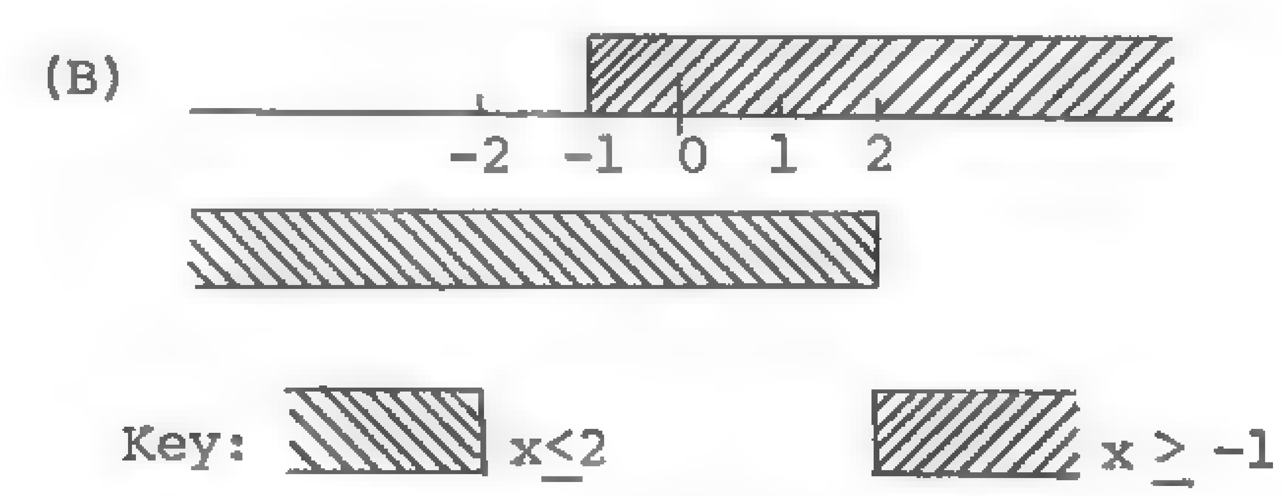
Thus $x \le -1 \cap x \ge 2 = \emptyset$

Case 2: $x - 2 \le 0$, $x + 1 \ge 0$.

Solving these two inequalities,

$x \le 2, \quad x \ge -1$

Graph these inequalities on number line (B).



The interval of x which satisfies both inequalities at the same time is $-1 \le x \le 2$. Note that the two inequalities intersect in this interval on number line (B), that is

$$x \ge -1 \cap x \le 2 = -1 \le x \le 2.$$

Hence, the solution to the inequality $x^2 - x - 2 \le 0$ is

the set:

$$\{x \mid -1 \leq x \leq 2\}$$

GRAPHING QUADRATIC EQUATIONS AND INEQUALITIES

• PROBLEM 8-32

> Draw the graphs of $f(x) = x^2$, $g(x) = 3x^2$, and also $h(x) = \frac{1}{2}x^2$ on one set of coordinate axes.

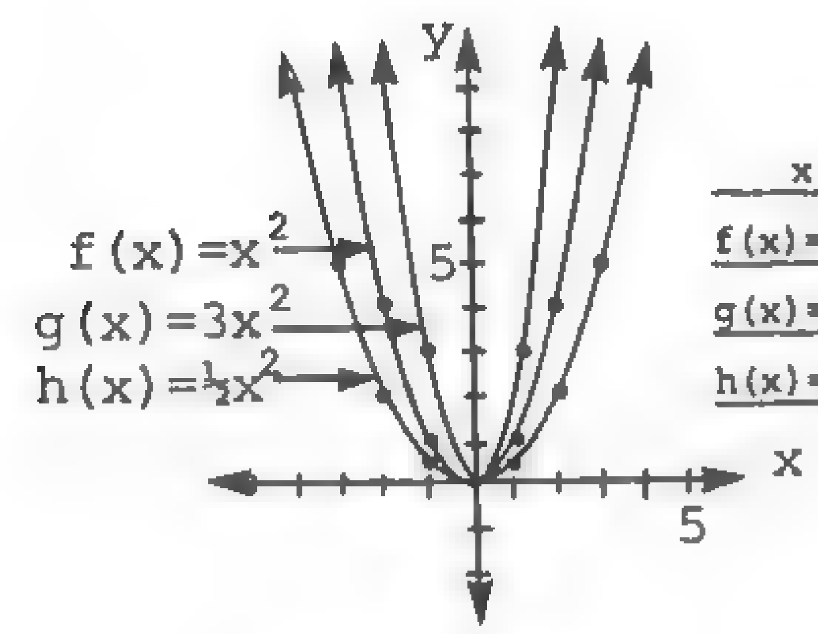
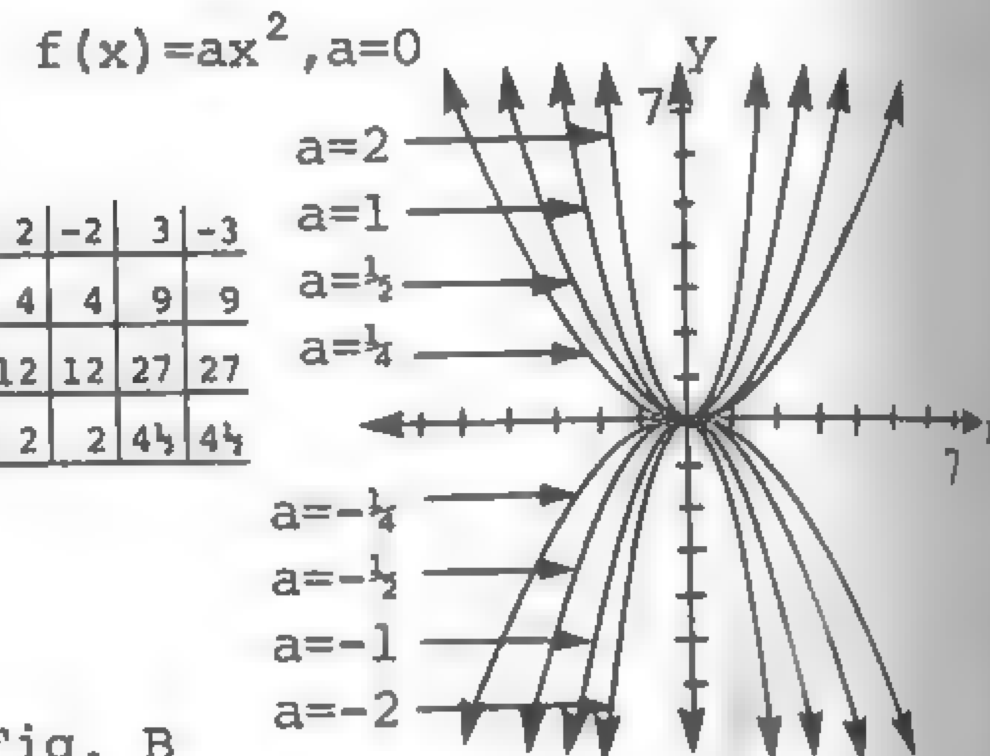


Fig. A

x	0	1	-1	2	-2	3	-3
$f(x)=x^2$	0	1	1	4	4	9	9
$g(x)=3x^2$	0	3	3	12	12	27	27
$h(x)=\frac{1}{2}x^2$	0	½	½	2	2	4½	4½

Fig. B



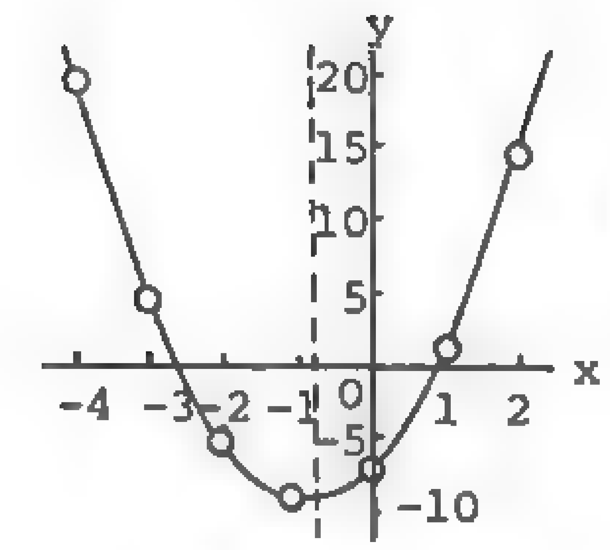
Solution: We construct a composite table showing the values of each function corresponding to selected values for x.

In the example, we graphed three instances of the function $f(x) = ax^2$, $a > 0$. For different values of a, how do the graphs compare? (Fig. A). Assigning a given value to a has very little effect upon the main characteristics of the graph. The coefficient a serves as a "stretching factor" relative to the y-axis. As a increases, the two branches of the curve approach the y-axis. The curve becomes "thinner". As a decreases, the curve becomes "flatter" and approaches the x-axis.

The graph of $f(x)^2 = ax$, $a \neq 0$, is called a parabola. (Fig. B). The point (0,0) is the vertex, or turning point, of the curve; the y-axis is the axis of symmetry. The value of a determines the shape of the curve. For $a > 0$ the parabola opens upward and for $a < 0$ the parabola opens downward.

• PROBLEM 8-33

> Graph the function $3x^2 + 5x - 7$.



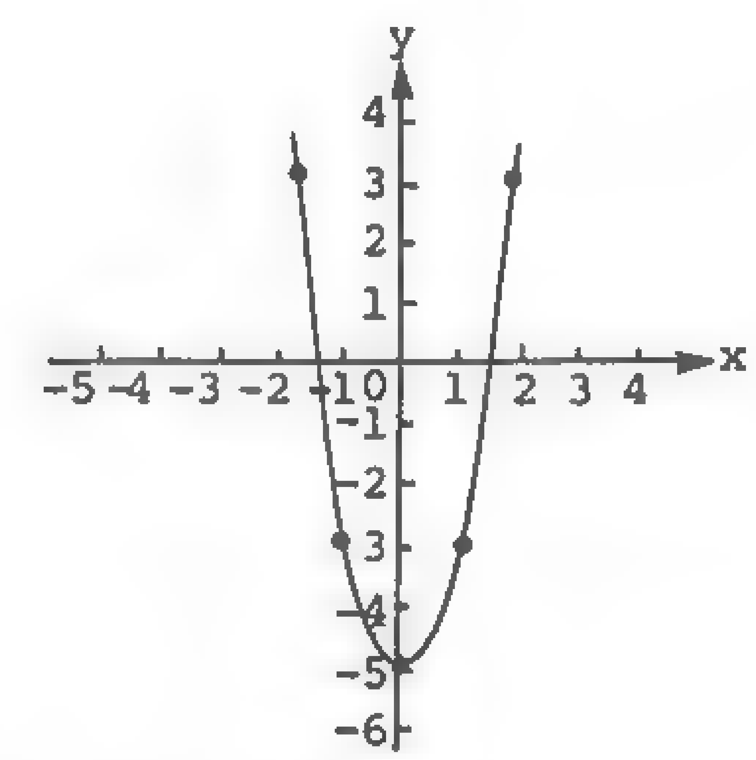
Solution: Let $y = 3x^2 + 5x - 7$. Substitute values for x and then find the corresponding values of y. This is done in the following table.

x	$y = 3x^2 + 5x - 7$
-4	21
-3	5
-2	-5
-1	-9
0	-7
1	1
2	15

These points are plotted and joined by a smooth curve in the figure.

• PROBLEM 8-34

> Graph $y = 2x^2 - 5$.



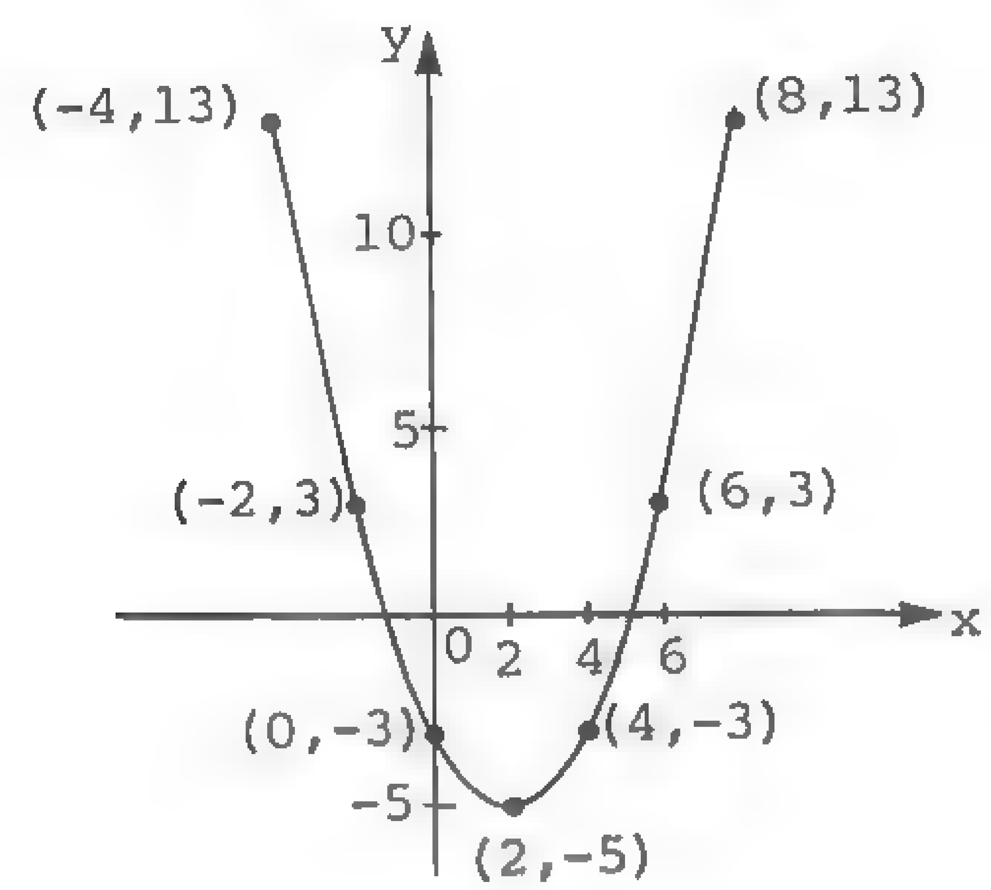
Solution: Graphs of the form $y = kx^2 + c$ are parabolas that stretch upward from a minimum point on the y-axis. From the table

x	-2	-1	0	1	2
y	3	-3	-5	-3	3

we obtain the graph shown in the accompanying figure.

• PROBLEM 8-35

Construct the graph of $\{(x,y) \mid y = \frac{1}{2}x^2 - 2x - 3\}$.



Solution: In general, the graph of the quadratic function $y = ax^2 + bx + c$ is a parabola. Some of its properties are:

1. If $a > 0$, the graph opens upward.
2. If $a < 0$, the graph opens downward.

We note that $a = \frac{1}{2}$ in the equation $y = \frac{1}{2}x^2 - 2x - 3$. Hence the graph is a parabola opening upward since $a > 0$. In order to determine the vertex (x_v, y_v) of the parabola, we complete the square by the following procedure:

Factor out $\frac{1}{2}$ from $y = \frac{1}{2}x^2 - 2x - 3$. Then $\frac{1}{2}x^2 - 2x - 3 = \frac{1}{2}(x^2 - 4x - 6)$ Complete the square of $x^2 - 4x$. This is done by taking $\frac{1}{2}$ of the coefficient of x and squaring it. Thus, the constant term is

$$[\tfrac{1}{2}(-4)]^2 = (-2)^2 = 4.$$

To keep the same equation we must retain -6. Thus, we express it as $4 - 10$. Then, $\frac{1}{2}(x^2 - 4x - 6) = \frac{1}{2}(x^2 - 4x + 4 - 10)$.

Factor $(x^2 - 4x + 4)$ into $(x-2)(x-2) = (x-2)^2$.

Then, $\frac{1}{2}(x^2 - 4x + 4 - 10) = \frac{1}{2}\left[(x-2)^2 - 10\right]$.

Now, $y = \frac{1}{2}\left[(x-2)^2 - 10\right]$. Since the parabola opens upward, the vertex is the value where y is minimum. $(x-2)^2$ will always be positive since the square of a positive or of a negative number is always positive. Therefore, the minimum value of y will occur when $(x-2) = 0$. Consequently y is least when $x-2 = 0$ or when $x = 2$. To find y, substitute $x = 2$ into $y = \frac{1}{2}\left[(x-2)^2 - 10\right]$.

$$\begin{aligned} y &= \tfrac{1}{2}\left[(2-2)^2 - 10\right] \\ &= \tfrac{1}{2}\left[0^2 - 10\right] \\ &= \tfrac{1}{2}[-10] \\ &= -5 . \end{aligned}$$

Hence, the vertex is $(2,-5)$.

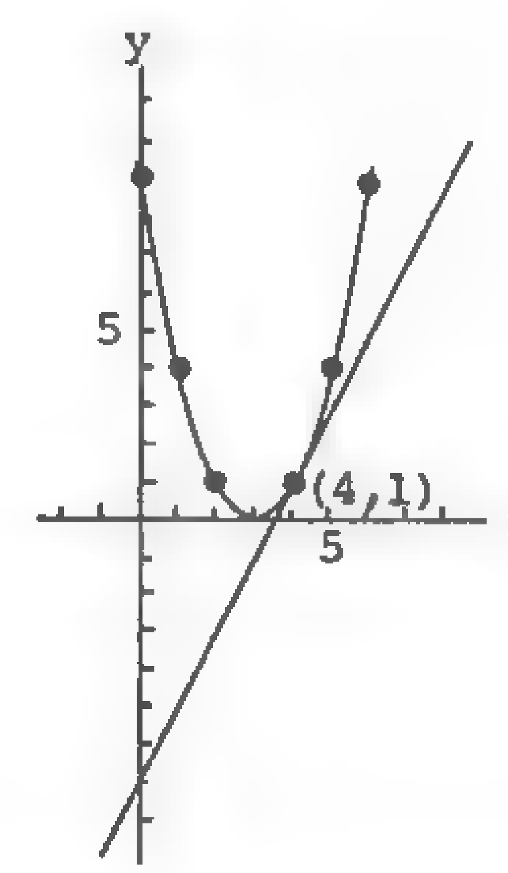
We now assign numbers to x and calculate the corresponding y-values to obtain points on the parabola. This is done in the following table:

x	$\frac{1}{2}x^2-2x-3$	y
-4	$\frac{1}{2}(-4)^2-2(-4)-3$	13
-2	$\frac{1}{2}(-2)^2-2(-2)-3$	3
0	$\frac{1}{2}(0)^2-2(0)-3$	-3
2	$\frac{1}{2}(2)^2-2(2)-3$	-5
4	$\frac{1}{2}(4)^2-2(4)-3$	-3
6	$\frac{1}{2}(6)^2-2(6)-3$	3
8	$\frac{1}{2}(8)^2-2(8)-3$	13

The graph is shown in the accompanying figure.

• PROBLEM 8-36

Find the equation of the tangent to the parabola $y = x^2 - 6x + 9$, if the slope of the tangent equals 2.



Solution: The equation of a straight line is $y = mx + k$ where m is the slope and k is the y-intercept. The equation

$$y = 2x + k \qquad (1)$$

represents a family of parallel lines with slope 2, some of which intersect the parabola in two points, others which have no point of intersection with the parabola, and just one which intersects the parabola in only one point. The problem is to find the value of k so that the graph of Equation 1 intersects the parabola in just one point. If we solve the system

$$y = 2x + k \qquad (1)$$

$$y = x^2 - 6x + 9 \qquad (2)$$

by substitution, we get for the first step

$$2x + k = x^2 - 6x + 9 \qquad \text{or}$$

$$x^2 - 8x + 9 - k = 0 \qquad (3)$$

This is a quadratic equation of the form $ax^2 + bx + c = 0$. The discriminant determines the nature of the roots when $ax^2 + bx + c = 0$. The condition that Equation 3 has but one solution is that the discriminant, $b^2 - 4ac$ equals 0. Therefore, if $a = 1$, $b = -8$, $c = 9 - k$, then $b^2 - 4ac = 64 - 4(9 - k) = 0$ or $k = -7$.

Substituting this value of k in equation 1, we have $y = 2x - 7$ which is the equation of the tangent to the given parabola when the slope of the tangent is equal to 2. The figure is the graph of the parabola and the tangent. The student may verify that the point of contact is (4, 1). This is shown by substituting (4, 1) into

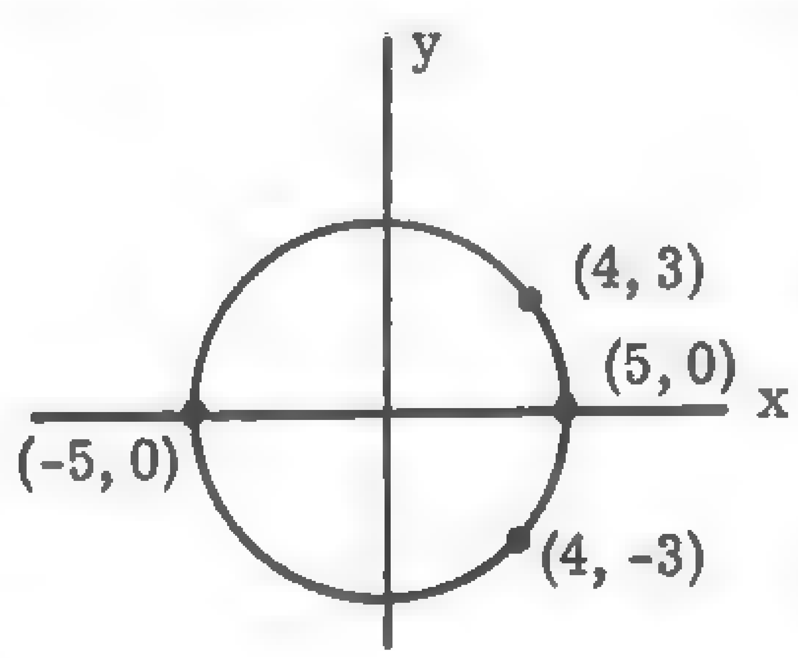
$$y = 2x - 7 = x^2 - 6x + 9$$

$$1 = 2(4) - 7 = 4^2 - 6(4) + 9$$

$$1 = \qquad 1 = 1$$

• PROBLEM 8-37

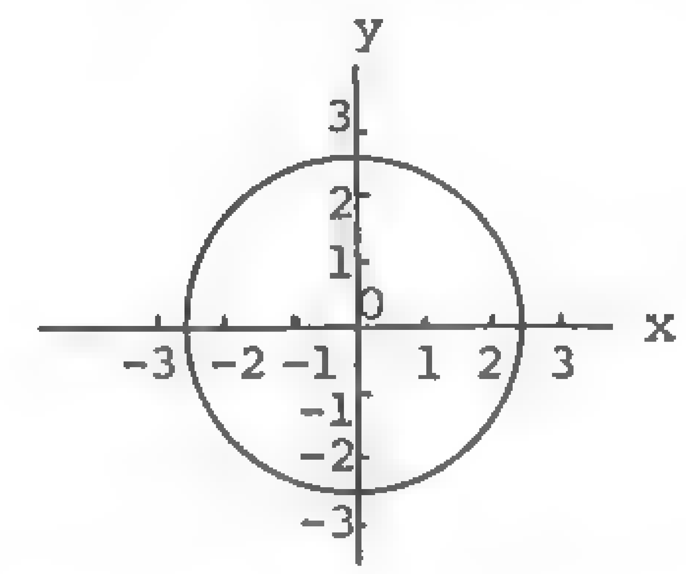
Discuss the graph of the equation $x^2 + y^2 = 25$.



Solution: This is an equation of the form $x^2 + y^2 = r^2$, and therefore its graph is a circle with radius $r = 5$ and center at the origin (see figure). Note that the graph does not represent a function since, except for $x = -5$ or $x = 5$, each permissible value of x is associated with two values of y. For example, for $x = 4$, we have the ordered pairs (4,3) and (4,-3). The domain of this function is $\{x \mid -5 \leq x \leq 5\}$. The range of this function is $\{y \mid -5 \leq y \leq 5\}$.

• PROBLEM 8-38

Graph the equation $2x^2 + 2y^2 - 13 = 0$.



Solution: In order to verify that this is the equation of a circle, we put the equation in the standard form: add 13 to both sides of the given equation.

$$2x^2 + 2y^2 - 13 + 13 = 0 + 13$$

$$2x^2 + 2y^2 = 13$$

Divide both sides of this equation by 2.

$$\frac{2x^2 + 2y^2}{2} = \frac{13}{2}$$

or

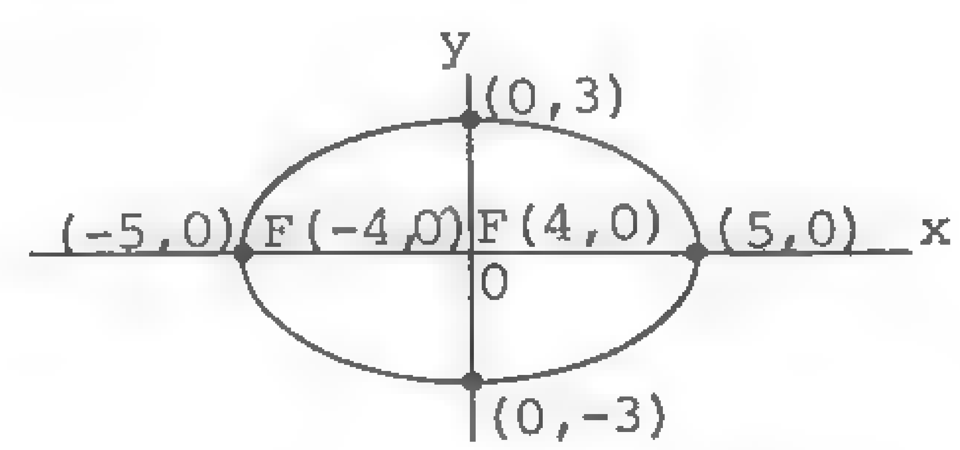
$$x^2 + y^2 = \frac{13}{2},$$ which is the standard form for the equation of a circle with its center at the origin (0,0) and

$$\text{radius} = r = \sqrt{\frac{13}{2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}\,\sqrt{2}}{\sqrt{2}\,\sqrt{2}} = \frac{\sqrt{26}}{2}.$$

Therefore, the radius of the circle is approximately $\frac{5.1}{2}$ or 2.55. The graph is represented in the Figure.

• PROBLEM 8-39

Discuss the graph of $\frac{x^2}{25} + \frac{y^2}{9} = 1$.



Solution: Since this is an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

with a = 5 and b = 3, it represents an ellipse. The simplest way to sketch the curve is to find its intercepts. If we set x = 0, then

$$y = \sqrt{\left[1 - \frac{x^2}{25}\right]9} = \sqrt{\left[1 - \frac{0^2}{25}\right]9} = \pm 3$$

so that the y-intercepts are at (0,3) and (0,-3). Similarly, the x-intercepts are found for y = 0:

$$x = \sqrt{\left[1 - \frac{y^2}{9}\right]25}$$

$$= \sqrt{\left[1 - \frac{0^2}{9}\right]25}$$

$$= \pm 5$$

to be at (5,0) and (-5,0) (see figure). To locate the foci we note that

$$c^2 = a^2 - b^2 = 5^2 - 3^2$$

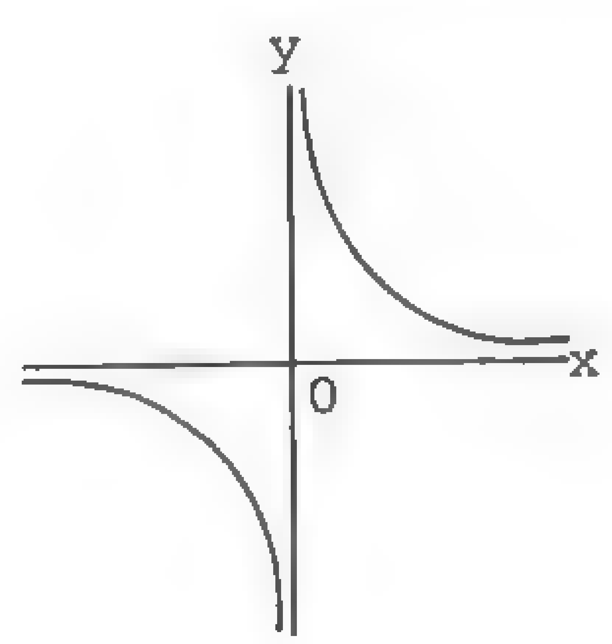
$$c^2 = 25 - 9 = 16$$

$$c = \pm 4.$$

The foci lie on the major axis of the ellipse. In this case it is the x-axis since a = 5 is greater than b = 3. Therefore, the foci are (±c,0), that is, at (-4,0) and (4,0). Therefore, the foci are at (-4,0) and (4,0). The sum of the distances from any point on the curve to the foci is 2a = 2(5) = 10.

• PROBLEM 8-40

Draw the graph of xy = 6.



Solution: Since the product is positive the values of x and y must have the same sign, that is, when x is positive y must also be

positive and when x is negative then y is also negative. Moreover, neither x nor y can be zero(or their product would be zero not 6), so that the graph never touches the coordinate axes. Solve for y and we obtain y = 6/x. Substituting values of x into this equation we construct the following chart:

x:	-6	-3	-2	-1	1	2	3	6
y:	-1	-2	-3	-6	6	3	2	1

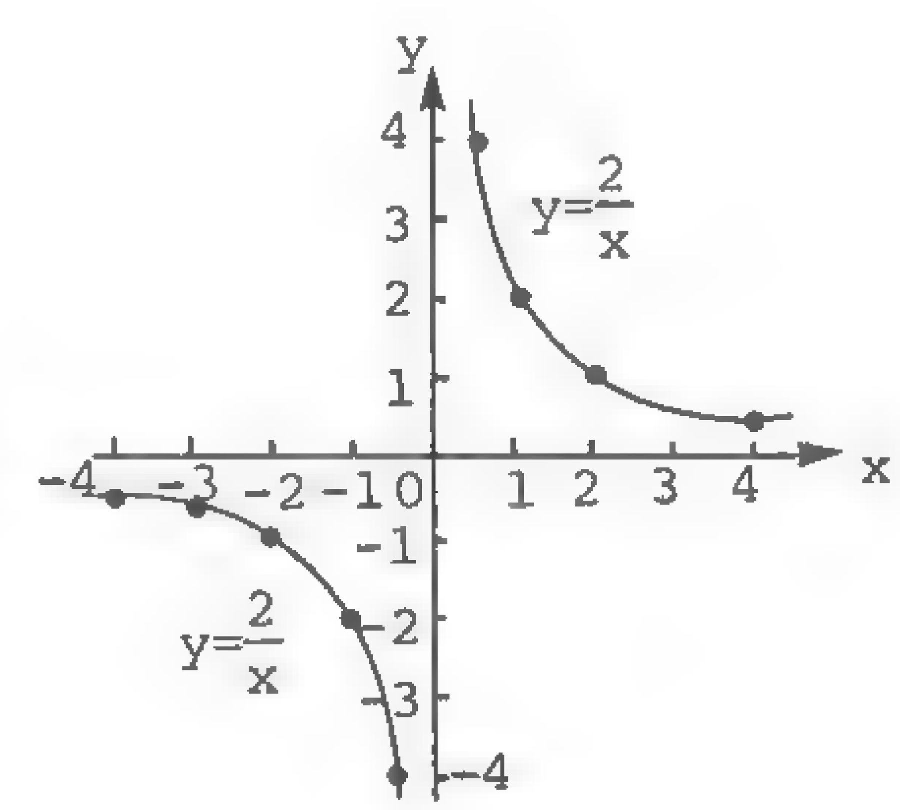
The graph is obtained by plotting the above points and then joining them with a smooth curve, remembering that the curve can never cross a coordinate axis. The graph of the equation, xy = k, is a hyperbola for all nonzero real values of k. If k is negative, then x and y must have opposite signs, and the graph is in the second and fourth quadrants as opposed to the first and third.

• PROBLEM 8-41

Sketch the graph of the equation y = 2/x.

Solution: Substitute values for x and then find the corresponding values for y. This is done in the following table.

x	$y = \frac{2}{x}$
-4	$-\frac{1}{2}$
-3	$-\frac{2}{3}$
-2	-1
-1	-2
$-\frac{1}{2}$	-4
$\frac{1}{2}$	4
1	2
2	1
3	$\frac{2}{3}$
4	$\frac{1}{2}$

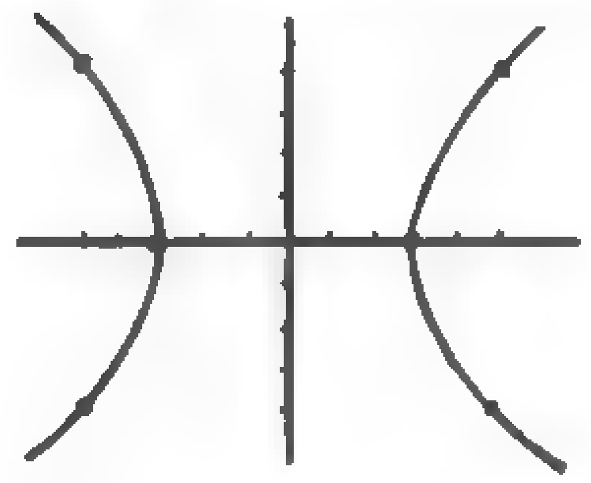


The graph is shown in the figure. This graph is an example of an equilateral hyperbola. Notice in the graph that, when x takes on larger and larger positive values, y gets closer and closer to 0. When x takes on larger and larger negative values, y also gets closer and closer to 0. Also, when x gets closer and closer to 0, y either takes on larger and larger positive values or larger and larger

negative values. Note also that x cannot be 0, since $y = \frac{2}{x} = \frac{2}{0}$ is not defined.

• PROBLEM 8-42

Discuss the graph of $\frac{x^2}{9} - \frac{y^2}{9} = 1$.



Solution: $\frac{x^2}{9} - \frac{y^2}{9} = 1$ is an equation of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a = 3$ and $b = 3$. Therefore the graph is a hyperbola. The x-intercepts are found by setting $y = 0$:

$$\frac{x^2}{9} - \frac{0^2}{9} = 1$$

$$x^2 = 9$$

$$x = \pm 3.$$

Thus, the x-intercepts are at (-3,0) and (3,0). There are no y-intercepts since for x = 0 there are no real values of y satisfying the equation, i.e., no real value of y satisfies

$$\frac{0^2}{9} - \frac{y^2}{9} = 1$$

$$y^2 = -9, \quad y = \sqrt{-9}.$$

Solving the original equation for y:

$$y = \sqrt{\left(1 - \frac{x^2}{9}\right)(-9)} \quad \text{or} \quad y = \sqrt{x^2 - 9}$$

shows that there will be no permissible values of x in the interval $-3 < x < 3$. Such values of x do not yield real values for y. For x = 5 and x = -5 use the equation for y to obtain the ordered pairs (5,4), (5,-4), (-5,4), and

(-5,-4) as indicated in the figure. The foci of the hyperbola are located at $(\pm c, 0)$, where

$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 3^2 = 9 + 9 = 18$$

$$c = \pm\sqrt{18} = \pm 3\sqrt{2}.$$

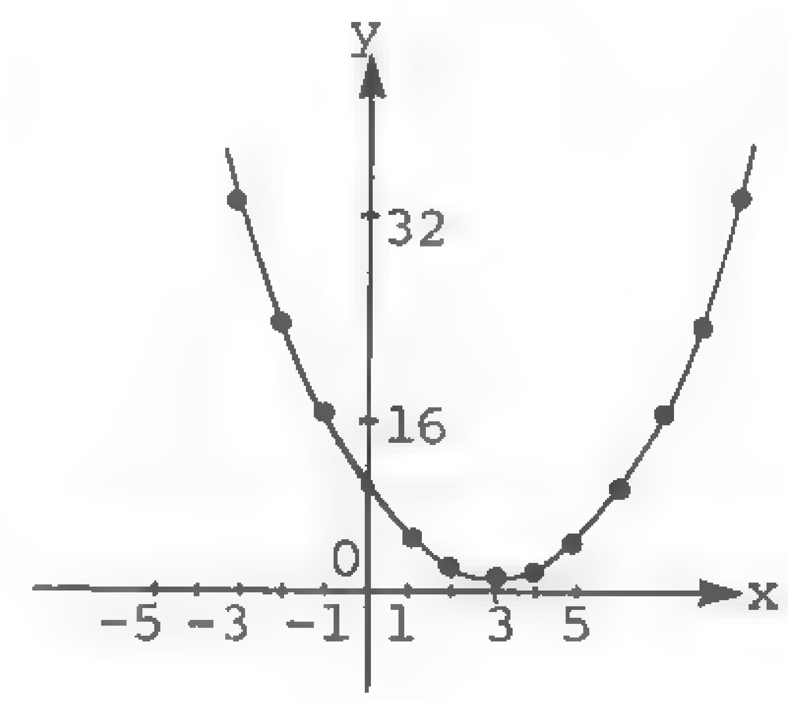
Therefore, the foci are at $(-3\sqrt{2}, 0)$ and $(3\sqrt{2}, 0)$.

• PROBLEM 8-43

Find the solution set of $x^2 - 6x + 10 > 0$ by the graphical method.

Solution: First we graph the function $y = x^2 - 6x + 10$. Assign values to x and then calculate y-values.

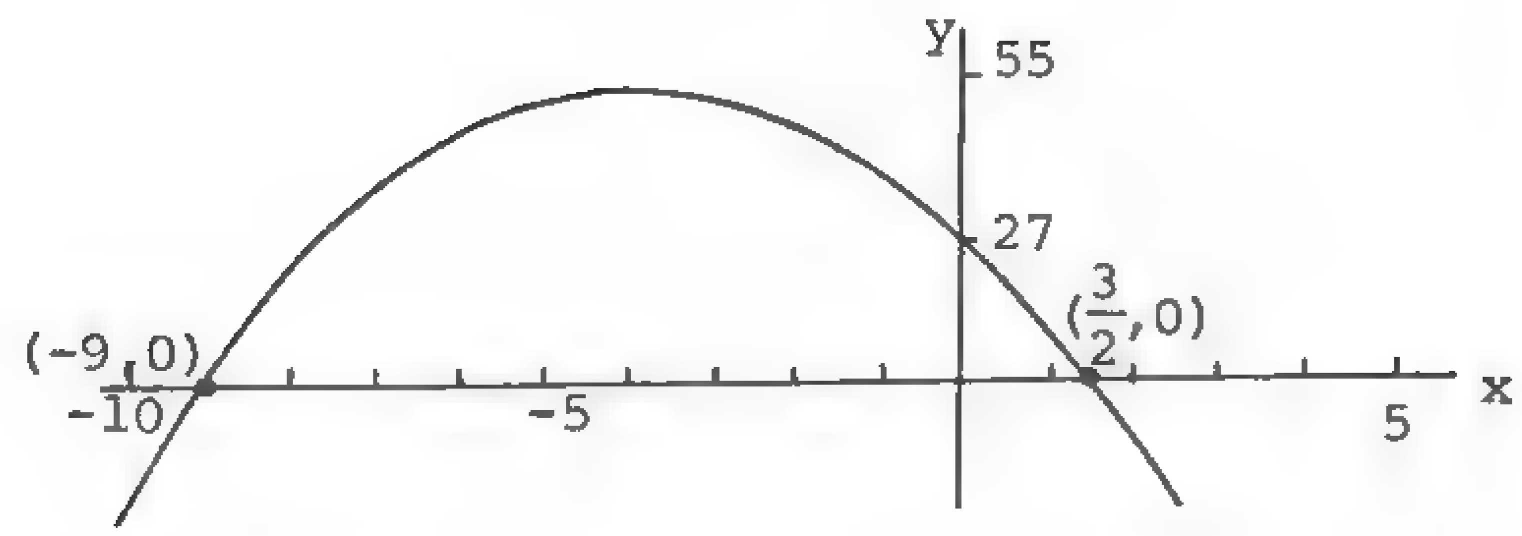
x	$x^2 - 6x + 10$	y
-3	$(-3)^2 - 6(-3) + 10$	37
-2	$(-2)^2 - 6(-2) + 10$	26
-1	$(-1)^2 - 6(-1) + 10$	17
0	$(0)^2 - 6(0) + 10$	10
1	$(1)^2 - 6(1) + 10$	5
2	$(2)^2 - 6(2) + 10$	2
3	$(3)^2 - 6(3) + 10$	1
4	$(4)^2 - 6(4) + 10$	2



See graph. The curve is the graph of $y = x^2 - 6x + 10$. Since the graph is entirely above the X axis, the solution set of $x^2 - 6x + 10 > 0$ is the set of all real numbers.

• PROBLEM 8-44

Solve the inequality $-2x^2 - 15x + 27 > 0$.



Solution: The graph of the corresponding quadratic function $y = -2x^2 - 15x + 27$ is a parabola which is concave down since the coefficient of the x^2-term is negative. Its zeros are the solution set of the quadratic equation $-2x^2 - 15x + 27 = 0$.

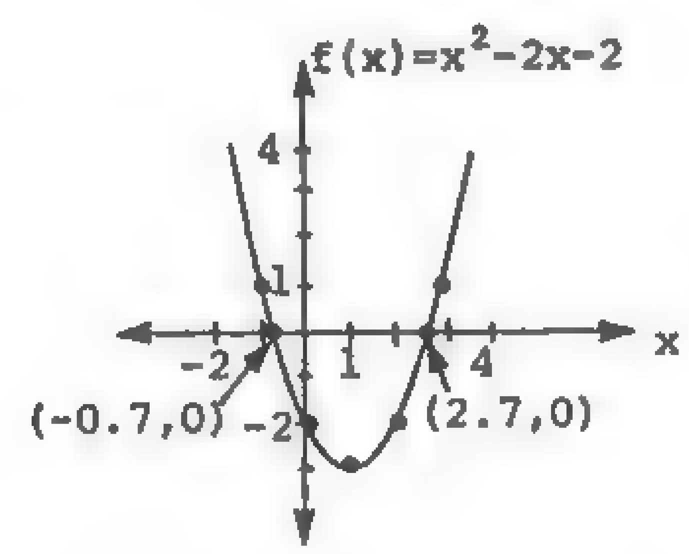
$(2x - 3)(-x - 9) = 0$ factor

$2x - 3 = 0$	$-x - 9 = 0$	set each factor = 0 to find all x's which make the product 0.
$x = \frac{3}{2}$	$x = -9$	

Therefore, the solution set of the given inequality is the set of all x's such that the graph of $-2x^2 - 15x + 27$ lies above the x-axis, i.e., $\{x \mid y = -2x^2 - 15 + 27 > 0\}$. This set is $\{x \mid -9 < x < \frac{3}{2}\}$. (see Figure).

● **PROBLEM** 8-45

Solve: $x^2 < 2x + 2$.



Solution: To solve the inequality $x^2 < 2x + 2$, transfer all the terms to one side. Subtracting $(2x+2)$ from both sides,

$$x^2 - 2x - 2 < 0 .$$

We shall solve this inequality graphically. The function that we are dealing with is $f(x) = x^2 - 2x - 2$. We must find the region where it is less than zero. Thus, we graph the left side as a function of x. Choose values of x and calculate the corresponding f(x) values, as shown in the following table:

x	x^2-2x-2	f(x)
-3	$(-3)^2-2(-3)-2 = 9+6-2$	13
-2	$(-2)^2-2(-2)-2 = 8-2$	6
-1	$(-1)^2-2(-1)-2 = 1+2-2$	1
0	$(0)^2-2(0)-2 = 0-0-2$	-2
1	$(1)^2-2(1)-2 = 1-2-2$	-3
2	$(2)^2-2(2)-2 = 4-4-2$	-2
3	$(3)^2-2(3)-2 = 9-6-2$	1

The graph of $f(x) = x^2-2x-2$ is shown in the accompanying figure. $f(x)<0$ when the curve lies below the x-axis, that is, for values of x between -0.7 and 2.7. (see figure) Hence, $-0.7<x<2.7$.

Since these results were read from the curve, they are only approximations. If the student should read 2.6 or 2.8 instead of 2.7, his result would be acceptable.

In case more accuracy is desired, we can solve the corresponding equation $x^2 - 2x - 2 = 0$ and find $x = 1 \pm \sqrt{3}$. Thus the curve crosses the x-axis when $x = 1 - \sqrt{3}$ and $x = 1 + \sqrt{3}$, and the inequality is true for $1 - \sqrt{3} < x < 1 + \sqrt{3}$. Note $x = 1 + \sqrt{3} \approx 2.7$ and $x = 1 - \sqrt{3} \approx -0.7$.

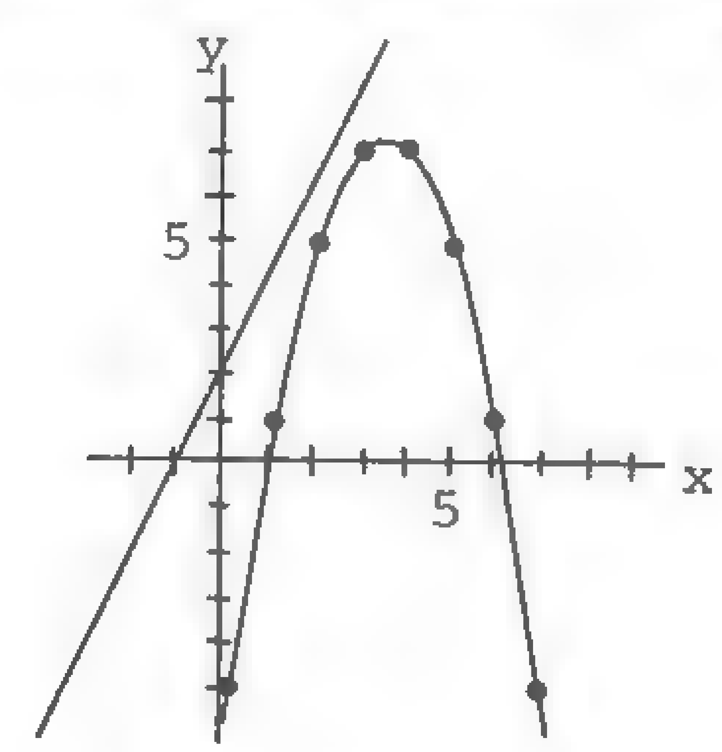
SYSTEMS OF QUADRATIC EQUATIONS

● **PROBLEM** 8-46

Solve the system

$$y = -x^2 + 7x - 5 \qquad (1)$$

$$y - 2x = 2 \qquad (2)$$



Solution: Solving Equation (2) for y yields an expression for y in terms of x. Substituting this expression in Equation (1),

$$2x + 2 = -x^2 + 7x - 5 \quad (3)$$

We have a single equation, in terms of a single variable, to be solved. Writing Equation (3) in standard quadratic form,

$$x^2 - 5x + 7 = 0 \quad (4)$$

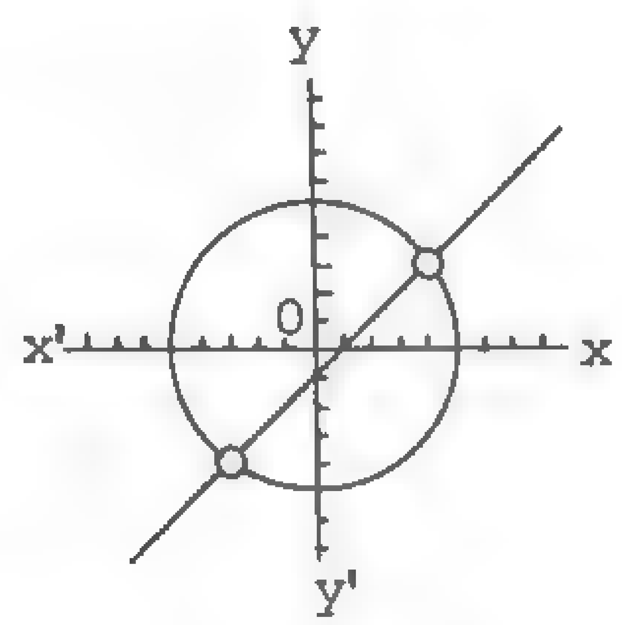
Since the equation is not factorable the roots are not found in this manner. Evaluating the discriminant will indicate whether Equation (4) has real roots. The discriminant, $b^2 - 4ac$, of Equation (4) equals $(-5)^2 - 4(1)(7) = 25 - 28 = -3$. Since the discriminant is negative, Equation (4) has no real roots, and therefore the system has no real solution. In terms of the graph, the figure shows that the parabola and the straight line have no point in common.

• PROBLEM 8-47

Solve the system

$$x^2 + y^2 = 25, \quad (1)$$

$$x - y = 1. \quad (2)$$



Solution: Solve (2) for y (the problem can be done similarly for x instead): The method of substitution is most easily employed in this example to solve the system.

$$y = x - 1. \quad (3)$$

Substitute x - 1 for y in (1):

$$x^2 + (x - 1)^2 = 25. \quad (4)$$

$$x^2 + x^2 - 2x + 1 = 25.$$

From (4) $\quad 2x^2 - 2x - 24 = 0,$

or $\quad x^2 - x - 12 = 0.$

Solve (5) by factoring: $\quad (x - 4)(x + 3) = 0$

$$x - 4 = 0 \quad x + 3 = 0$$

$$x = 4 \text{ or } -3.$$

Substituting 4 for x in (2), we obtain

$$4 - y = 1 \text{ or } y = 3.$$

Substituting -3 for x in (2), we obtain

$$-3 - y = 1 \text{ or } y = -4.$$

This gives $\begin{matrix} x = 4 \\ y = 3 \end{matrix}$ and $\begin{matrix} x = -3 \\ y = -4 \end{matrix}$ for the solutions.

Check:

for x = 4, y = 3

in Eq. (1): $(4)^2 + (3)^2 = 25$

$$16 + 9 = 25$$

$$25 = 25$$

in Eq. (2): $(4) - (3) = 1$

$$1 = 1$$

for x = -3, y = -4

in Eq. (1): $(-3)^2 + (-4)^2 = 25$

$$9 + 16 = 25$$

$$25 = 25$$

in Eq. (2): $(-3) - (-4) = 1$

$$-3 + 4 = 1$$

$$1 = 1.$$

Graphical meaning of the two solutions. We may plot the graph for each of the equations (1) and (2). The graph of

$$x - y = 1$$

is the straight line shown in the figure, and the graph of

$$x^2 + y^2 = 25$$

is the circle there shown. To draw the graph of (1), the student may give various values to x and calculate the corresponding values for y from $y = \pm\sqrt{25 - x^2}$.

Any point on the straight line (2) has coordinates that satisfy Equation (2). Any point on the circle (1) has coordinates that satisfy Equation (1). The points (4,3) and (-3,-4) lie on both graphs and satisfy both Equations (1) and (2). That is to say, each point of intersection of the graph of (1) with the graph of (2) gives a pair of numbers that is a solution of the system.

• PROBLEM 8-48

Use the graphical method to find the simultaneous solution set of

$$x^2 + x - 2 > 0$$

and $$\frac{3}{4}x + \frac{3}{2} < 0$$

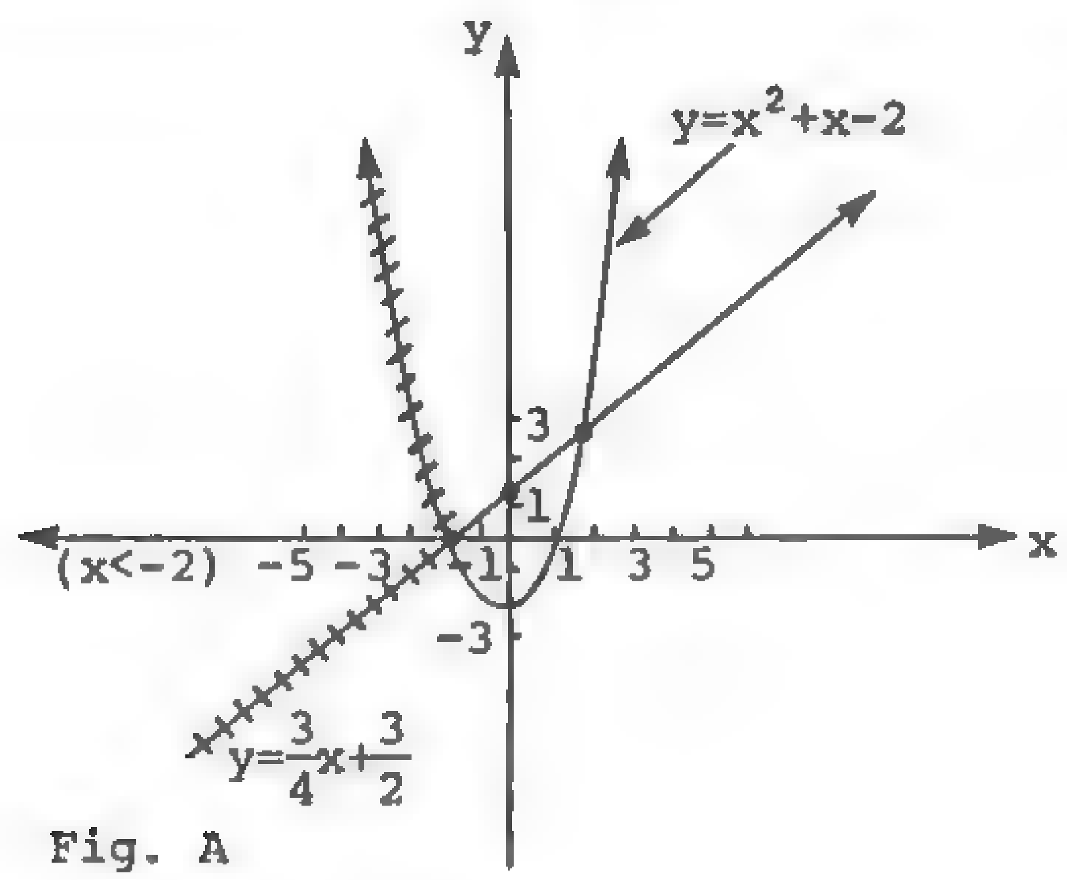
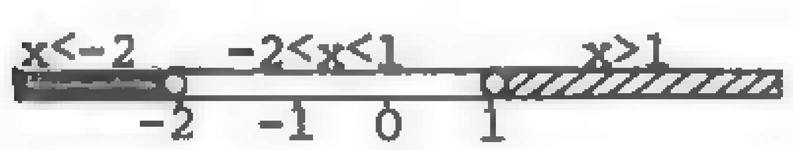


Fig. A



The slashed lines represent the simultaneous solution set.

Fig. B

<u>Solution:</u> We construct the graphs of

$$y = x^2 + x - 2 \text{ and } y = \frac{3}{4}x + \frac{3}{2}$$

Set up tables for both equations to calculate y values.

For $y = x^2 + x - 2$

x	$x^2 + x - 2$ =	y
− 4	$(-4)^2 + (-4) - 2$	10
− 3	$(-3)^2 + (-3) - 2$	4
− 2	$(-2)^2 + (-2) - 2$	0
− 1	$(-1)^2 + (-1) - 2$	-2
0	$(0)^2 + (0) - 2$	-2
1	$(1)^2 + (1) - 2$	0
2	$(2)^2 + 2 - 2$	4
3	$(3)^2 + 3 - 2$	10

For $y = \frac{3}{4}x + \frac{3}{2}$

x	$\frac{3}{4}x + \frac{3}{2}$ =	y
-2	$\frac{3}{4}(-2) + \frac{3}{2}$	0
0	$\frac{3}{4}(0) + \frac{3}{2}$	$\frac{3}{2}$
2	$\frac{3}{4}(2) + \frac{3}{2}$	3

See Figure A for graphs.

Now, find the region where the inequality $x^2 + x - 2 > 0$ holds. The function $f(x) = x^2 + x - 2$ can be factored into $(x + 2)(x - 1)$. Set $f(x) = 0$ and find the roots of this equation. Here $x = -2$ and $x = 1$. Mark the roots on the x-axis and consider the regions into which the roots divide the x-axis. They are $x < -2$, $-2 < x < 1$, and $x > 1$. See Figure B.

For each of these regions, choose a value of x and investigate the algebraic signs of the factors of the function f(x). Then look at the sign of their product, f(x). The table summarizes the process.

	$f(x) = (x + 2)(x - 1)$		
Regions	$x < -2$	$-2 < x < 1$	$x > 1$
x-value	$x = -3$	$x = 0$	$x = 2$
Factors of f(x)	(x+2)(x-1)	(x+2)(x − 1)	(x+2)(x-1)
Signs of factors	(-) (-)	(+) (-)	(+) (+)
∴	$f(x) > 0$	$f(x) < 0$	$f(x) > 0$

For our problem we now know that the graph of $x^2 + x - 2$ is greater than zero, that is, above the x-axis, for $x < -2$ or $x > 1$.

Call the second function g(x). Thus, $g(x) = \frac{3}{4}x + \frac{3}{2}$. We are interested in finding the values of x for which the function g(x) is negative or when it is below the x-axis. Therefore,

$$\frac{3}{4}x + \frac{3}{2} < 0 \Leftrightarrow \frac{3}{4}x < -\frac{3}{2} \Leftrightarrow 3x < -6 \Leftrightarrow x < -2$$

Hence, the solution set is $\{x \mid x < -2\}$. That is, the graph of $y = \frac{3}{4}x + \frac{3}{2}$ lies below the x- axis when $x<-2$.

Simultaneously we solve for x and we obtain $\{x \mid x < -2\}$. See Figure A.

CHAPTER 9

EXPANSIONS AND SEQUENCES

PROGRESSIONS AND SEQUENCES

• PROBLEM 9-1

If the 6th term of an arithmetic progression is 8 and the 11th term is -2, what is the 1st term? What is the common difference?

Solution: An arithmetic progression is a sequence of numbers where each term excluding the first is obtained from the preceding one by adding a fixed quantity to it. This constant amount is called the common difference. Let a = value of first term,and d = common difference

Term of sequence:	1^{st}	2^{nd}	3^{thrd}	4^{th}	...	n^{th}
Value of term:	a	$a+d$	$a+2d$	$a+3d$	...	$a+(n-1)d$

Use the formula for the nth term of the sequence to write equations for the given 6th and 11th terms, to determine a and d.

11th term: $a + (11 - 1)d = -2$

6th term: $a + (6 - 1)d = 8$. Simplifying the above equations we obtain:

$$a + 10d = -2 \qquad (1)$$
$$a + 5d = 8 \qquad (2)$$

$5d = -10$ Subtracting (2) from (1)

$d = -2$ Substituting in (1)

$$a + 10(-2) = -2$$
$$a = 18$$

The first term is 18 and the common difference is -2.

● PROBLEM 9-2

Find a_n for the sequence 1, 4, 7, 10,...

Solution: An arithmetic progression (A.P.) is a sequence of numbers each of which, after the first, is obtained by adding to the preceding number a constant number, d, called the common difference. Thus 1, 4, 7, 10,... is an arithmetic progression because each term is obtained by adding 3 to the preceding number. The nth term, a_n, of an A.P. is:

$$a_n = a_1 + (n-1)d$$

where a_1 = first term of the progression; d = common difference; n = number of terms.

Thus, with $a_1 = 1$ and $d = 3$,

$$a_n = 1 + (n-1)3 = 1 + 3n - 3$$

$$a_n = 3n - 2$$

It is easily verified by substitution that $a_n = 3n - 2$ will suffice, i.e.

$$a_1 = 3(1) - 2 = 1$$
$$a_2 = 3(2) - 2 = 4$$
$$a_3 = 3(3) - 2 = 7$$
$$a_4 = 3(4) - 2 = 10$$
$$a_5 = 3(5) - 2 = 13 = a_4 + d = 10 + 3$$

by definition of an A.P. etc.

● PROBLEM 9-3

Find the first term of an arithmetic progression if the fifth term is 29 and d is 3.

Solution: The n^{th} term, or last term, of an arithmetic progression (A.P.) is:

$$\ell = a_1 + (n-1)d \qquad (1)$$

where a_1 = first term of the progression
d = common difference
n = number of terms
$\ell = n^{th}$ term, or last term.

Using this formula we can find the first term of an A.P. whose fifth term is 29 and d is 3. Since $\ell = a_5 = 29$, $d = 3$, and $n = 5$, substituting into equation (1) gives:

$$29 = a_1 + (5 - 1)3$$
$$29 = a_1 + 12$$
$$a_1 = 29 - 12 = 17$$

Thus, the first term is 17.

● PROBLEM 9-4

Given that the first term of an arithmetic sequence is 56 and the seventeenth term is 32, find the tenth term and the twenty-fifth term.

Solution: The formula for the nth term, or last term, of an arithmetic sequence is:

$$\ell = s_1 + (n-1)d$$

where s_1 = first term of the sequence
d = common difference
n = number of terms
ℓ = nth term, or last term.

Since we are given the first and seventeenth term of the sequence we can use this information, with n = 17, to find d before we proceed to find the tenth and twenty-fifth terms. Thus we find d by

$$s_{17} = s_1 + (17 - 1)d,$$
$$32 - 56 = 16d$$
$$16d = -24$$
$$d = -\frac{3}{2}$$

Now we can find the tenth term, s_{10}, as

$$s_{10} = 56 + (10 - 1)\left(-\frac{3}{2}\right) = 56 - \frac{27}{2} = \frac{85}{2}$$

and the twenty-fifth term, s_{25}, as

$$s_{25} = 56 + (25 - 1)\left(-\frac{3}{2}\right) = 56 - 36 = 20$$

● PROBLEM 9-5

If the first term of an arithmetic progression is 7, and the common difference is -2, find the fifteenth term and the sum of the first fifteen terms.

Solution: An arithmetic progression is a sequence of numbers each of which is obtained from the preceding one by adding a constant quantity to it, the common difference, d. If we designate the first term by a and the common difference by d, then the terms can be expressed as follows:

terms of series	1	2	3	.....	n
value of term	a	a + d	a + 2d		$\ell = a + (n-1)d$

In this example a = 7, and d = -2. To find the fifteenth term, we have n = 15. The nth term is $a + (n-1)d$. For n = 15, $a + (n-1)d = 7 + (15 - 1)(-2) = 7 - 28 = -21$. To find the sum of the first fifteen terms apply the following formula:

$$S_n = \frac{n}{2}(a + \ell)$$

$$S_{15} = \frac{15}{2}[7 + (-21)] = \frac{15}{2}(-14) = -105.$$

• **PROBLEM 9-6**

If the first term of a geometric progression is 9 and the common ratio is $-\frac{2}{3}$, find the first five terms.

Solution: A geometric progression (G.P.) is a sequence of numbers each of which, after the first, is obtained by multiplying the preceding number by a constant number called the common ratio, r. Thus a G.P. such as a_1, a_2, a_3, a_4, a_5, ... or a_1, a_1r, a_2r, a_3r, a_4r, ... with $a_1 = 9$ and $r = -\frac{2}{3}$ is determined as follows:

$$a_1 = 9$$
$$a_2 = 9\left(-\frac{2}{3}\right) = -6$$
$$a_3 = (-6)\left(-\frac{2}{3}\right) = 4$$
$$a_4 = 4\left(-\frac{2}{3}\right) = -\frac{8}{3}$$
$$a_5 = \left(-\frac{8}{3}\right)\left(-\frac{2}{3}\right) = \frac{16}{9}$$

Thus, the first five terms are: 9, -6, 4, $-\frac{8}{3}$, $\frac{16}{9}$

• **PROBLEM 9-7**

Find the next three terms of the geometric progression 1,2,4, 8,... .

Solution: First determine the common ratio of the progression by dividing any term by the term immediately preceeding it. Once the common ratio has been determined any term can be computed by multiplying the term immediately preceeding the unknown term by the common ratio. The common ratio is 2 so the 5th term is 8 X 2 = 16. The 6th term is 16 X 2 = 32. The 7th term is 32 X 2 = 64. The first seven terms of the geometric progression are:

1,2,4,8,16,32,64

• **PROBLEM 9-8**

The seventh term of a geometric progression is 192 and r = 2. Find the first four terms.

Solution: The formula for the nth term, or last term, of a geometric progression is:

$$\ell = s_1 r^{n-1}$$

where s_1 = first term of the progression
r = common ratio
n = number of terms
ℓ = nth term, or last term

Since we are given the seventh term and the common ratio of the progression we can use this information, with n = 7, to find the first term:

$$s_7 = s_1 r^{n-1}$$

$$192 = s_1(2)^{7-1} = 2^6 s_1 = 64 s_1$$

$$s_1 = \frac{192}{64} = 3$$

Then, since a geometric progression is a sequence of numbers each of which, after the first, is obtained by multiplying the preceding number by a constant number called the common ratio,

$s_1 = 3$, $s_2 = 3 \cdot 2 = 6$, $s_3 = 6 \cdot 2 = 12$, and $s_4 = 12 \cdot 2 = 24$

FACTORIAL NOTATION

• **PROBLEM 9-9**

Find the value of $\frac{7!}{5!}$.

Solution: Apply the definition of factorial. If n is any positive integer, the symbol n! is the product of the

integers from 1 up to and including n.

$$n! = 1 \cdot 2 \cdot 3 \ldots n.$$

$$\frac{7!}{5!} = \frac{(1)(\cancel{2})(\cancel{3})(\cancel{4})(\cancel{5})(6)(7)}{\cancel{1}(\cancel{2})(\cancel{3})(\cancel{4})(\cancel{5})}$$

$$= 6(7)$$

$$= 42$$

• PROBLEM 9-10

Simplify the expression $\frac{5! - 4!}{6!}$.

Solution: Since $5! = 5 \cdot 4!$ and $6! = 6 \cdot 5 \cdot 4!$, the expression $\frac{5! - 4!}{6!}$ becomes:

$$\frac{5! - 4!}{6!} = \frac{(5 \cdot 4!) - 4!}{6 \cdot 5 \cdot 4!}$$

$$= \frac{(4! \cdot 5) - 4!}{4! \cdot 6 \cdot 5} \qquad (1)$$

By the distributive property, which states that $ab + ac = a(b+c)$, equation (1) becomes:

$$\frac{5! - 4!}{6!} = \frac{\cancel{4!}(5-1)}{\cancel{4!} \cdot 6 \cdot 5}$$

$$= \frac{5-1}{6 \cdot 5} = \frac{4}{6 \cdot 5}$$

$$= \frac{4}{30} .$$

Thus,

$$\frac{5! - 4!}{6!} = \frac{2}{15} .$$

• PROBLEM 9-11

If n and r are positive integers, and $r < n$, show that

$$n! = r!(r + 1)(r + 2) \cdot \ldots \cdot n.$$

Solution: By definition of factorial,

$$r! = 1 \cdot 2 \cdot \ldots \cdot r .$$

Then,

$$r!(r+1)(r+2) \cdot \ldots \cdot n = (1 \cdot 2 \cdot \ldots \cdot r)(r+1)(r+2) \cdot \ldots \cdot n$$

$$r!(r+1)(r+2) \cdot \ldots \cdot n = 1 \cdot 2 \cdot \ldots \cdot n \qquad (1)$$

Again, by definition of factorial,

$$n! = 1 \cdot 2 \cdot \ldots \cdot n .$$

Hence, equation (1) becomes:

$$r!(r+1)(r+2) \cdot \ldots \cdot n = n!$$

or

$$n! = r!(r+1)(r+2) \cdot \ldots \cdot n .$$

COMBINATIONS

• PROBLEM 9-12

Find ${}_9C_4$.

Solution: By definition, combinations of b different things taken a at a time, ${}_bC_a = \frac{b!}{a!(b-a)!}$; hence by substitution, ${}_9C_4 = \frac{9!}{4!(9-4)!}$.

$$= \frac{9!}{4!5!}$$

$$= \frac{9.8.7.6.(5.4.3.2.1)}{4.3.2.1.(5.4.3.2.1)}$$

Cancelling $5! = (5.4.3.2.1)$ out of numerator and denominator, we multiply to obtain:

$${}_9C_4 = \frac{3024}{24} = 126 .$$

• PROBLEM 9-13

Evaluate each of the following symbols:
(a) C(6,3) (b) C(18,16)

Solution: Recalling the general formula for the number of combinations of n different things taken r at a time

$$C(n,r) = \frac{n!}{r!(n-r)!}:$$

(a) $$C(6,3) = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{\overset{2}{\cancel{6}} \cdot 5 \cdot \cancel{4} \cdot \cancel{3!}}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{3!}} = 20.$$

(b) $$C(18,16) = \frac{18!}{16!(18-16)!} = \frac{18!}{16!2!} = \frac{\overset{9}{\cancel{18}} \cdot 17 \cdot \cancel{16!}}{\cancel{2} \cdot 1 \cdot \cancel{16!}} = 153$$

• PROBLEM 9-14

Find the binomial expansion of $(2x - 5)^4$.

Solution: The generalized form of the binomial expansion is

$$(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + {}_nC_3 a^{n-3}b^3 + \dots + {}_nC_{n-1} a^1 b^{n-1} + {}_nC_n a^0 b^n$$

Here we take $a = 2x$, $b = -5$, and $n = 4$.

$$(2x-5)^4 = (2x)^4 + {}_4C_1(2x)^3(-5)^1 + {}_4C_2(2x)^2(-5)^2 + {}_4C_3(2x)^1(-5)^3 + {}_4C_4(2x)^0(-5)^4$$

$$= 16x^4 + {}_4C_1\, 8x^3\cdot(-5) + {}_4C_2\, 4x^2(25) + {}_4C_3\, 2x(-125) + {}_4C_4 1(-5)^4$$

Note that ${}_nC_r = \frac{n!}{r!(n-r)!}$. Therefore

$$(2x-5)^4 = 16x^4 + \frac{4!}{1!3!}8x^3(-5) + \frac{4!}{2!2!}(4x^2)(25) + \frac{4!}{3!1!}(2x)(-125) + \frac{4!}{4!0!}(1)(-5)^4$$

$$= 16x^4 + \frac{4\cdot\cancel{3!}}{\cancel{3!}}8x^3(-5) + \frac{\overset{2}{\cancel{4}}\cdot 3\cdot\cancel{2!}}{\underset{1}{\cancel{2}}\cdot 1\ \cancel{2!}}(100x^2) + \frac{4\cdot\cancel{3!}}{\cancel{3!}\cancel{1!}}(-250x) + 625$$

$$= 16x^4 - 160x^3 + 600x^2 - 1,000x + 625.$$

• PROBLEM 9-15

Find the expansion of $(a - 2x)^7$.

Solution: Use the binomial formula:

$$(u+v)^n = u^n + nu^{n-1}v + \frac{n(n-1)}{2}u^{n-2}v^2 + \frac{n(n-1)(n-2)}{2\cdot 3}u^{n-3}v^3 + \dots + v^n$$

and substitute a for u and (-2x) for v and 7 for n to obtain:

$$(a-2x)^7 = [a+(-2x)]^7$$

$$= a^7 + 7a^6(-2x) + \frac{7\cdot\overset{3}{\cancel{6}}}{\underset{1}{\cancel{2}}}a^5(-2x)^2 + \frac{7\cdot\cancel{6}\cdot 5}{\cancel{2}\cdot\cancel{3}}a^4(-2x)^3$$

$$+ \frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}}a^3(-2x)^4 + \frac{7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}}a^2(-2x)^5$$

$$+ \frac{7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot\cancel{6}}a^1(-2x)^6 + \frac{\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot\cancel{6}\cdot\cancel{7}}a^0(-2x)^7$$

$$(a-2x)^7 = a^7 - 14a^6x + 84a^5x^2 - 280a^4x^3 + 560a^3x^4 - 672a^2x^5 + 448a\,x^6 - 128x^7.$$

• PROBLEM 9-16

Expand $(x + 2y)^5$.

Solution: Apply the binomial theorem. If n is a positive integer, then

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n.$$

Note that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and that $0! = 1$. Then, we obtain:

$$(x+2y)^5 = \binom{5}{0}x^5(2y)^0 + \binom{5}{1}x^4(2y)^1 + \binom{5}{2}x^3(2y)^2 + \binom{5}{3}x^2(2y)^3 + \binom{5}{4}x^1(2y)^4 + \binom{5}{5}x^0(2y)^5$$

$$= \frac{\cancel{5!}}{0!\cancel{5!}}x^5 + \frac{5!}{1!4!}x^4 2y + \frac{5!}{2!3!}x^3(4y^2) + \frac{5!}{3!2!}x^2(8y^3) + \frac{5!}{4!1!}x(16y^4) + \frac{5!}{5!0!}1(32y^5)$$

$$= x^5 + \frac{5\cdot\cancel{4!}}{\cancel{4!}}x^4 2y + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}}{\underset{1}{\cancel{2}}\ 1\ \cancel{3!}}x^3(4y^2)$$

$$+ \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}}{\cancel{3!}\ \underset{1}{\cancel{2}}\cdot 1}x^2(8y^3) + \frac{5\cdot\cancel{4!}}{\cancel{4!}1!}x(16y^4) + \frac{\cancel{5!}}{\cancel{5!}0!}(32y^5)$$

$$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5.$$

Find the fifth term of $\left(2 + 2x^3\right)^{17}$.

Solution: Use the Binomial Theorem which states that

$$(c+b)^n = \frac{1}{0!}c^n + \frac{n}{1!}c^{n-1}b + \frac{n(n-1)}{2!}c^{n-2}b^2 + \ldots + ncb^{n-1} + b^n .$$

Replacing c by a and b by $2x^3$:

$$\left(a+2x^3\right)^{17} = \frac{1}{0!}a^{17} + \frac{17}{1!}a^{16}\left(2x^3\right)^1 + \frac{17\cdot16}{2!}a^{15}\left(2x^3\right)^2$$

$$+ \frac{17\cdot16\cdot15}{3!}a^{14}\left(2x^3\right)^3 + \frac{17\cdot16\cdot15\cdot14}{4!}a^{13}\left(2x^3\right)^4$$

$$+ \ldots$$

The fifth term of this expansion is:

$$\frac{17\cdot16\cdot15\cdot14}{4!}a^{13}\left(2x^3\right)^4 = \frac{17\cdot\overset{4}{\cancel{16}}\cdot\overset{5}{\cancel{15}}\cdot\overset{7}{\cancel{14}}}{\underset{1}{\cancel{4}}\cdot\underset{1}{\cancel{3}}\cdot\underset{1}{\cancel{2}}\cdot1}a^{13}\left(2^4\right)\left(x^3\right)^4$$

$$= \frac{17\cdot4\cdot5\cdot7}{1}\ a^{13}\ 16x^{12}$$

$$= 38{,}080\ a^{13}\ x^{12}$$

four terms of the expansion:

$$(2-1)^{\frac{1}{2}} = \sqrt{2} - \tfrac{1}{2}\left(\frac{1}{2^{\frac{1}{2}}}\right) - \frac{1}{8}\left(\frac{1}{2^{\frac{3}{2}}}\right) - \frac{3}{48}\left(\frac{1}{2^{\frac{5}{2}}}\right) + \ldots$$

$$= \sqrt{2} - \tfrac{1}{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{8}\left(\frac{1}{\sqrt{2^3}}\right) - \frac{3}{48}\left(\frac{1}{\sqrt{2^5}}\right) + \ldots$$

$$= \sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{8}\left(\frac{1}{\sqrt{2^2}\sqrt{2}}\right) - \frac{3}{48}\left(\frac{1}{\sqrt{2^2}\sqrt{2^2}\sqrt{2}}\right) + \ldots$$

$$= \sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{8}\left(\frac{1}{2\sqrt{2}}\right) - \frac{3}{48}\left(\frac{1}{(2)(2)\sqrt{2}}\right) + \ldots$$

$$= \sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{16\sqrt{2}} - \frac{3}{48}\left(\frac{1}{4\sqrt{2}}\right) + \ldots$$

$$= \sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{16\sqrt{2}} - \frac{3}{192}\ \frac{1}{\sqrt{2}} + \ldots$$

$$= \sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{16\sqrt{2}} - \frac{1}{64}\ \frac{1}{\sqrt{2}} + \ldots$$

$$= \sqrt{2} - \frac{1}{2\sqrt{2}} - \frac{1}{16\sqrt{2}} - \frac{1}{64\sqrt{2}} + \ldots$$

Since $\sqrt{2} = 1.414$,

$$(2-1)^{\frac{1}{2}} = 1.414 - \frac{1}{2(1.414)} - \frac{1}{16(1.414)} - \frac{1}{64(1.414)}$$

$$= 1.414 - \frac{1}{2.828} - \frac{1}{22.624} - \frac{1}{90.496}$$

$$(2-1)^{\frac{1}{2}} = 1.414 - 0.354 - 0.044 - 0.011 \qquad (1)$$

Hence, the right side of equation (1) represents the first four terms of the expansion of $(2-1)^{\frac{1}{2}}$.

Note that the sum of these four terms is 1.005.

CHAPTER 10

LOGARITHMS AND EXPONENTIALS

• PROBLEM 10-1

Write $5^3 = 125$ in logarithmic form.

Solution: The statement $b^y = x$ is equivalent to the statement $\log_b x = y$, where b is the base and y is the exponent. The latter form is the logarithmic form. Thus $5^3 = 125$ in logarithmic form is $\log_5 125 = 3$, where the base is 5 and the logarithm of 125 is 3.

• PROBLEM 10-2

Find $\log_{10} 100$.

Solution: The following solution presents 2 methods for solving the given problem.

Method I. The statement $\log_{10} x = y$ is equivalent to $10^y = x$, hence $\log_{10} 100 = x$ is equivalent to $10^x = 100$. Since $10^2 = 100$, $\log_{10} 100 = 2$.

Method II. Note that $100 = 10 \times 10$; thus $\log_{10} 100 = \log_{10} (10 \times 10)$. Recall: $\log_x (a \times b) = \log_x a + \log_x b$, therefore

$$\begin{aligned}\log_{10} (10 \times 10) &= \log_{10} 10 + \log_{10} 10 \\ &= 1 + 1 \\ &= 2.\end{aligned}$$

• PROBLEM 10-3

Find the logarithm of 3^2.

Solution: Recall that $\log_b x^y = y \log_b x$; thus

$$\log_{10} 3^2 = 2 \log_{10} 3$$

Referring to a table of common logarithms we find:

$$\begin{aligned}\log_{10} 3 = .4771; \text{ hence, } & \\ &= 2(.4771) \\ &= .9542 .\end{aligned}$$

Thus, $\log_{10} 3^2 = .9542$.

We could have obtained the same result by noting that $3^2 = 9$. Using a log table to find $\log_{10} 9$, we observe:

$$\log_{10} 9 = .9542, \text{ as above.}$$

• PROBLEM 10-4

Write $\frac{1}{2} = \log_9 3$ in exponential form.

Solution: The statement $\log_b x = y$ is equivalent to the statement $b^y = x$, where b is the base and y the exponent. The latter form is the exponential form. Thus, $\frac{1}{2} = \log_9 3$ in exponential form is $9^{\frac{1}{2}} = 3$, where the base is 9 and the exponent is $\frac{1}{2}$.

• PROBLEM 10-5

If $\log_3 N = 2$, find N.

Solution: The equation $\log_x a = y$ is equivalent to the equation $x^y = a$. Thus $\log_3 N = 2$ is equivalent to the equation $3^2 = N$. $3^2 = 9$, hence $N = 9$.

• PROBLEM 10-6

Find the value of x if $\log_4 64 = x$.

Solution: $\log_b u = v$

is equivalent to,

$$b^v = u,$$

thus the exponential equivalent of

$\log_4 64 = x$ is $4^x = 64$.

Since,

$$4^3 = 4 \cdot 4 \cdot 4 = 64$$

$$\log_4 64 = 3.$$

That is,

$$x = 3.$$

• PROBLEM 10-7

Find $\log_3 729$.

Solution: Since we are working with log base 3, we check whether 729 has factors of 3.

$$729 = 3 \cdot 243 = 3 \cdot (3 \cdot 81) = 3 \cdot [3 \cdot (3 \cdot 27)] = 3[3 \cdot 3(3 \cdot 9)]$$
$$= [3 \cdot 3 \cdot 3 \cdot 3 \cdot (3 \cdot 3)]$$
$$= 3^6$$

Thus $\log_3 729 = \log_3 3^6 = 6$ (because $\log_b b^a = a$ is equivalent to the statement $b^a = b^a$).

Check: $\log_3 729 = 6$ is equivalent to $3^6 = 729$ which we have just seen to be true.

• PROBLEM 10-8

Find the value of N if $\log_8 N = \frac{2}{3}$.

Solution: The inverse of the logarithmic function, $y = \log_a N$, is the exponential function $N = a^y$. Then for

$$\log_8 N = \frac{2}{3},$$

$$N = 8^{2/3} = \sqrt[3]{(8)^2} = (\sqrt[3]{8})^2 = (2)^2 = 4$$

$N = 4.$

• PROBLEM 10-9

Express the logarithm of 7 to the base 3 in terms of common logarithms.

Solution: By definition, if $\log_b a = n$, then $b^n = a$. Therefore, if $\log_3 7 = x$, then $3^x = 7$. Take the logarithm of both sides:

$$\log 3^x = \log 7.$$

By the law of the logarithm of a power of a positive number which states that $\log a^n = n \log a$, $\log 3^x = x \cdot \log 3$. Hence, $x \log 3 = \log 7$. Divide both sides of this equation by $\log 3$:

$$\frac{x \cancel{\log 3}}{\cancel{\log 3}} = \frac{\log 7}{\log 3}.$$

Therefore, $x = \frac{\log 7}{\log 3} = \log_3 7$ is the logarithm of 7 to the base 3 expressed in terms of common logarithms.

• PROBLEM 10-10

What is the value of b in the relation $\log_b \frac{1}{25} = -2$?

Solution: Since the statement,

$$\log_b x = y$$

is equivalent to,

$$b^y = x, \quad \log_b \frac{1}{25} = -2$$

is equivalent to

$$b^{-2} = \frac{1}{25}. \quad x^{-y} = \frac{1}{x^y}$$

thus,

$b^{-2} = \frac{1}{b^2}$. Therefore $b^{-2} = \frac{1}{25}$ is equivalent to $\frac{1}{b^2} = \frac{1}{25}$.

Cross multiply to obtain the equivalent equation,

$$b^2 = 25.$$

Take the square root of both sides. Thus,

$$b = \pm 5$$

• PROBLEM 10-11

Express the logarithm of $\dfrac{\sqrt{a^3}}{c^5 b^2}$ in terms of log a, log b and log c.

Solution: We apply the following properties of logarithms:

$$\log_b(P\cdot Q) = \log_b P + \log_b Q$$

$$\log_b(P/Q) = \log_b P - \log_b Q$$

$$\log_b(P^n) = n \log_b P$$

$$\log_b(\sqrt[n]{P}) = \frac{1}{n} \log_b P$$

Therefore,

$$\log \frac{\sqrt{a^3}}{c^5 b^2} = \log \frac{a^{3/2}}{c^5 b^2}$$

$$= \log a^{3/2} - \log(c^5 b^2)$$

$$= 3/2 \log a - (\log c^5 + \log b^2)$$

$$= 3/2 \log a - \log c^5 - \log b^2$$

$$= 3/2 \log a - 5 \log c - 2 \log b .$$

• PROBLEM 10-12

If $\log_{10} 3 = .4771$ and $\log_{10} 4 = .6021$, find $\log_{10} 12$.

Solution: Since $12 = 3 \times 4$,

$$\log_{10} 12 = \log_{10}(3)(4).$$

Since $\log_b(xy) = \log_b x + \log_b y$

$$\log_{10}(3 \times 4) = \log_{10} 3 + \log_{10} 4$$

$$= .4771 + .6021$$

$$= 1.0792.$$

Thus $\log_{10} 12 = 1.0792.$

• PROBLEM 10-13

Given $\log_{10}2 = 0.3010$, find $\log_{10}32$.

Solution: Note that,

$$32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5.$$

Thus,

$$\log_{10}32 = \log_{10}2^5.$$

Recall the logarithmic property,

$$\log_b x^y = y \log_b x.$$

Hence,

$$\log_{10}32 = \log_{10}2^5 = 5 \log_{10}2$$

$$= 5\ (0.3010)$$

$$= 1.5050$$

• PROBLEM 10-14

If $\log_b 2 = .69$ and $\log_b 3 = 1.10$, find $\log_b 6$ and $\log_b 8$.

Solution: $6 = 2 \times 3$, therefore $\log_b 6 = \log_b(2 \times 3)$. Since

$$\log_a(y \times z) = \log_a y + \log_a z$$

$$\log_b(2 \times 3) = \log_b 2 + \log_b 3$$

Thus

$$\log_b 6 = \log_b 2 + \log_b 3 = .69 + 1.10 = 1.79.$$

$8 = 2^3$, therefore $\log_b 8 = \log_b 2^3$. Since

$$\log_a y^z = z \log_a y$$

$$\log_b 2^3 = 3 \log_b 2$$

thus

$$\log_b 8 = 3 \log_b 2 = 3(.69) = 2.07.$$

• PROBLEM 10-15

Given that $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$, find $\log_{10}\sqrt{6}$.

Solution: $\sqrt{6} = 6^{\frac{1}{2}}$, thus $\log_{10}\sqrt{6} = \log_{10}6^{\frac{1}{2}}$. Since $\log_b x^y = y \log_b x$,

$\log_{10}6^{\frac{1}{2}} = \frac{1}{2}\log_{10}6$. Therefore $\log_{10}\sqrt{6} = \frac{1}{2}\log_{10}6$. $6 = 3\cdot 2$, hence $\frac{1}{2}\log_{10}6 = \frac{1}{2}\log_{10}(3\cdot 2)$. Recall $\log_{10}(a\cdot b) = \log_{10}a + \log_{10}b$. Thus $\frac{1}{2}\log_{10}(3\cdot 2) = \frac{1}{2}\log_{10}3 + \log_{10}2$. Replace our values for $\log_{10}3$ and $\log_{10}2$,

$$= \tfrac{1}{2}(0.4771 + 0.3010)$$
$$= \tfrac{1}{2}(0.7781)$$
$$\approx 0.3890$$

Therefore $\log_{10}\sqrt{6} = 0.3890$.

• PROBLEM 10-16

Find the logarithm of 30,700.

Solution: First express 30,700 in scientific notation. $30{,}700 = 3.07 \times 10^4$. 4 is the characteristic. To find the mantissa, see a table of common logarithms of numbers The mantissa is 4871. Thus log 30,700 = 4 + .4871 = 4.4871.

• PROBLEM 10-17

Find log 0.0364 .

Solution: $0.0364 = 3.64 \times 10^{-2}$. Therefore, the characteristic, the power of 10, is -2. From a table of logarithms, the mantissa for 3.64 is 0.5611. Therefore, log 0.0364 = -2 + 0.5611 = -1.4389.

• PROBLEM 10-18

Find N if log N = 0.7917 - 3.

Solution: Using a table of logarithms, the mantissa .7917 is found to correspond to the number 6.19. Therefore the antilogarithm is 6.19. Then, since the characteristic is -3,

$$N = 6.19 \times 10^{-3} = 0.00619.$$

• PROBLEM 10-19

What is the value of log 0.0148?

Solution: $0.0148 = 1.48 \times 10^{-2}$. The characteristic is the exponent of 10. Hence, the characteristic is -2. The mantissa for 148 can be found in a table of logarithms. The mantissa is 0.1703. Therefore, log 0.0148 = -2 + 0.1703 = -2.0000 + 0.1703 = -1.8297. Notice that the number 0.0148 is less than 1. Therefore, the value of log 0.0148 must be negative, as it was found to be.

• PROBLEM 10-20

Determine x when log x = 3.1818.

Solution: $\log_{10} x = 3.1818$ is equivalent to $10^{3.1818} = x$. Since $a^{x+y} = a^x \cdot a^y$, $x = 10^{3.1818} = 10^3 \cdot 10^{.1818}$ $= 1{,}000 \cdot 10^{.1818}$.

We look in the body of the log table for the mantissa 0.1818 and find it in row 15 and column 2, so that the digits of x are 1.52. Thus

$$x = 1{,}000 \times 1.52$$
$$= 1520.$$

• PROBLEM 10-21

Find Antilog_{10} 0.8762 - 2 .

Solution: Let $N = \text{Antilog}_{10}\ 0.8762 - 2$. The following relationship between log and antilog exists: $\log_{10}x = a$ is the equivalent of $x = \text{antilog}_{10}a$. Therefore,

$$\log_{10}N = 0.8762 - 2.$$

The characteristic is -2. The mantissa is 0.8762. The number that corresponds to this mantissa is 7.52. This number is found from a table of common logarithms, base 10. Therefore,

$$N = 7.52 \times 10^{-2}$$
$$= 7.52 \times \left(\frac{1}{10^2}\right)$$
$$= 7.52 \times \left(\frac{1}{100}\right)$$
$$= 7.52(.01)$$
$$N = 0.0752 .$$

Therefore, $N = \text{Antilog}_{10}\ 0.8762 - 2 = 0.0752$.

• PROBLEM 10-22

Find the antilogarithm of 1.4349 to three significant figures.

Solution: The mantissa 4349 does not appear in a table of four place logarithms. However, it falls between the two listed mantissas:

4346, whose antilogarithm is 2.72

and 4362, whose antilogarithm is 2.73

Since the given mantissa is closer in value to the mantissa 4346, the three figure sequence of digits in the antilogarithm is 2.72. Using the characteristic 1, we have $2.72 \times 10^1 = 27.2$. Note that the characteristic is positive, (1), and is one less than the number of digits to the left of the decimal point. Thus there are two digits to the left of the decimal point, (27).

• PROBLEM 10-23

Find $\sqrt[5]{.2}$.

Solution: It is easier to perform this computation using logarithms.

Let $N = (.2)^{1/5}$. Then

$$\log N = \log (.2)^{1/5}$$

The characteristic of the common logarithm of any positive number less than 1 is negative and is one more than the number of zeros between the decimal point and the first digit.

To find the mantissa, see a table of common logarithms of numbers.

$$\text{Thus for } \log(.2) = \underbrace{.3010}_{\text{mantissa}} \overbrace{-1}^{\text{characteristic}}$$

$$\text{Then: } \log N = \frac{1}{5}\log .2 = \frac{.3010 - 1}{5}.$$

If we proceed with the arithmetic at this point, we shall find that $\log N = \frac{-.6990}{5} = -.1398$. But this last number must be written in standard form before we can solve for N. It is easier to replace the number (.3010 - 1) with its equivalent expression (4.3010 - 5) before we divide by 5. Then

$$\log N = \frac{4.3010 - 5}{5} = .8602 - 1.$$

See a table to find the mantissa, 8602 and look for the corresponding number. We find 7248. Adjust the decimal point by its characteristic, - 1. Hence,

$$\sqrt[5]{.2} = .7248.$$

• PROBLEM 10-24

Find $\sqrt[5]{20}$.

Solution: Let $N = \sqrt[5]{20} = 20^{1/5}$. Then, taking the logarithm of both sides:

$$\log N = \log 20^{1/5};$$

and since $\log a^x = x \log a$, $= 1/5 \log 20$. Using a table of logs of numbers, we find that the mantissa for 20 is .3010; and the characteristic is 1, since for a number greater than 1 (in this case 20) the characteristic is positive and one less than the number of digits before the decimal. Thus, we have:

$$\log N = \frac{1.3010}{5} = .2602.$$

To find N look up .2602 in a Table of Mantissas of Common Logarithms. We find the closest number is 182. Since we have 0.2602, which has a characteristic of zero, then there is one digit to the left of the decimal point. Thus, we adjust the decimal point and N = 1.82. Thus,

$$N = \sqrt[5]{20} = 1.82.$$

• PROBLEM 10-25

Evaluate $\dfrac{542.3\sqrt{0.1383}}{32.72}$ using logarithms.

Solution: Let x denote the above expression. Then,

$$x = \frac{542.3\sqrt{0.1383}}{32.72}.$$

Take the logarithms of both sides to obtain:

$$\log x = \log \frac{542.3(0.1383)^{1/2}}{32.72}$$

Apply the following properties of logarithms:

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\log(a\cdot b) = \log a + \log b$$

$$\log a^{n} = n \log a$$

Then,

$$\log x = \log 542.3(0.1383)^{1/2} - \log 32.72$$
$$= \log 542.3 + \log(0.1383)^{1/2} - \log 32.72$$
$$= \log 542.3 + \frac{1}{2}\log(0.1383) - \log 32.72$$

To find the characteristic and mantissa of each number, express each number in powers of 10. The exponent is the corresponding characteristic

$$542.3 = 5.423 \times 10^{2}$$
$$0.1383 = 1.383 \times 10^{-1}$$
$$32.72 = 3.272 \times 10^{1}$$

Note that these numbers are expressed as numbers between 1 and 10, multiplied by a power of ten. Look up the corresponding number in a table of common logarithms of numbers. This is the mantissa. Then,

$$\log 542.3 = 2.7343$$

$$\log 0.1383 = -1.1408.$$

The form 19.1408 - 20 is more convenient for computation.

$$\log 32.72 = 1.5148$$

Hence

$$\tfrac{1}{2}\log 0.1383 = \tfrac{1}{2}(19.1408 - 20) = 9.5704 - 10,$$
$$\log 542.3 = 2.7342$$
$$\tfrac{1}{2}\log 0.1383 + \log 542.3 = 9.5704 - 10 + 2.7342 = 2.3046$$
$$\log 32.72 = 1.5148$$
$$\log 542.3 + \tfrac{1}{2}\log(0.1383) - \log 32.72 = 2.3046 - 1.5148 = 0.7898$$
$$\log x = 0.7898$$

Look up the mantissa and we find the number is 6163. To adjust the decimal point, we note that the characteristic is 0. Since the characteristic is positive, it is one less than the number of digits to the left of the decimal point. Here, there is one digit to the left of the decimal point. Thus, the number is 6.163, and

$$x = 6.163.$$

Therefore,

$$\frac{542.3\sqrt{0.1383}}{32.72} = 6.163.$$

• PROBLEM 10-26

Use linear interpolation to find log 5.723.

Solution: Since 5.723 is .3 of the way from 5.72 to 5.73, we argue that log 5.723 is approximately .3 of the way from log 5.72 to log 5.73.

This is the basic idea involved in linear interpolation. We obtain log 5.72 and log 5.73 from a table of

common logarithms, and find the mantissas to be 7574 and 7582, respectively. We now use interpolation to find the mantissa for 5.723.

		Number	Mantissa		
1	.3	5.72	7574	x	8
		5.723			
		5.73	7582		

Note: Observe that 5.73 - 5.72 = 0.1, and 5.723 - 5.72 = .003; but we can rewrite these as 1 and .3 by shifting the decimal two places.

$$\frac{.3}{1} = \frac{x}{8}$$

$$x = 2.4 \approx 2$$

Thus, the mantissa of the given number is 7574 + 2 = 7576. Since the number is 5.723, and there is one digit before the decimal point, we know that the characteristic is one less than the number of digits, or one less than one, or 0. Thus,

$$\log 5.723 = .7576.$$

• PROBLEM 10-27

Find log 0.7056.

Solution: First determine the characteristic by realizing that it will be one more than the number of zeros to the right of the decimal point, with a negative sign because the number is less than one. Thus, the characteristic is - 1. To compute the mantissa notice that the number 7056 lies between 7050 and 7060, so that its log will occur between the logs of those numbers. Interpolating we obtain

			log		
10		7050	8482		6
	4	7056	?	x	
		7060	8488		

$$\frac{4}{10} = \frac{x}{6} \quad \text{or} \quad 10x = 24$$
$$x = 2.4$$

Now subtract this value from the higher mantissa

$$\begin{array}{r} 8488 \\ 2.4 \\ \hline 8485.6 \end{array}$$

The mantissa is always less than one so the decimal point must be moved four places to the left.

Now - 1 can be written as 9 - 10 for convenience, so our final answer becomes

$$\log 0.7056 = 9.84856-10$$

• PROBLEM 10-28

Determine log 51.83.

Solution: With the fourth digit dropped, the remaining three-digit number is 518, and the next larger one is 519. Schematically, we have

$$\begin{array}{cc} \text{Number} & \text{Log} \end{array}$$
$$10\left[\begin{array}{l} 3\left[\begin{array}{l} 5180 \\ 5183 \end{array}\right. \\ \quad 5190 \end{array}\right. \quad \left.\begin{array}{r} \left.\begin{array}{r} 7143 \\ x \end{array}\right] h \\ 7152 \end{array}\right] 9,$$

We set up the proportion $\frac{h}{9} = \frac{1}{10}$. Cross multiplying we obtain

$$10h = 27$$

$$h = \frac{27}{10} \simeq 3.$$

Thus $x = 7143 + 3 = .7146$.

Since $51.83 = 5.183 \times 10$

$$\begin{aligned} \log 51.83 &= \log(5.183 \times 10) \\ &= \log 5.183 + \log 10 \\ &= .7146 + 1 \\ &= 1.7146. \end{aligned}$$

• PROBLEM 10-29

Find log 513.06

Solution: First determine the characteristic according to the rule that the characteristic is one less than the number of digits to the left of the decimal point. In this case it is two. Now find the mantissa by checking the mantissa for the number 51300 which is 7101 and the mantissa for 51400 which is 7110; the mantissa for 51306 will lie between these two mantissas. Now a proportion can be set up to determine the actual mantissa for the number.

$$\begin{array}{cc} & \text{log} \end{array}$$
$$100\left[\begin{array}{l} \quad 51300 \\ 94\left[\begin{array}{l} 51306 \\ 51400 \end{array}\right. \end{array}\right. \qquad \left.\begin{array}{r} 7101 \\ \left.\begin{array}{r} ? \\ 7110 \end{array}\right] x \end{array}\right] 9$$

$\frac{94}{100} = \frac{x}{9}$, or cross-multiplying to obtain

$$(.94)9 = x = 8.46$$

Now subtract this value from the higher mantissa to obtain:

$$\begin{array}{r} 7110.00 \\ -\quad 8.46 \\ \hline 7101.54 \end{array}$$

Since the mantissa must be less than one, the decimal must be moved four places to the left, and the final answer is

$$\begin{aligned} \log 513.06 &= \text{characteristic} + \text{mantissa} \\ &= 2 + 0.71015 \\ &= 2.71015 \end{aligned}$$

• PROBLEM 10-30

Using a table of logarithms, determine the logarithm of 14.57.

Solution: This logarithm can be found by interpolation.

$$\begin{array}{cc} x & \log x \end{array}$$
$$.010\left[\begin{array}{l} .007\left[\begin{array}{l} 1.450 \\ 1.457 \end{array}\right. \\ \quad 1.460 \end{array}\right. \qquad \left.\begin{array}{r} \left.\begin{array}{r} .1614 \\ \log 1.457 \end{array}\right] d \\ .1644 \end{array}\right] .0030$$

Set up the following proportion:

$$\frac{.007}{.010} = \frac{d}{.0030}$$

Cross multiply to obtain

$$d = (.0030)\left(\frac{.007}{.010}\right)$$

$$= (3 \times 10^{-3})\left(\frac{7 \times 10^{-3}}{1 \times 10^{-2}}\right) = \frac{21 \times 10^{-6}}{1 \times 10^{-2}} = 21 \times 10^{-4}$$

$$= (21)(0.0001)$$

$$d = 0.0021$$

Hence, $\log 1.457 = 0.1614 + 0.0021$

$$= 0.1635$$

Note $14.57 = 1.457 \times 10^1$. The characteristic is the exponent of 10, thus the characteristic is 1, and $\log 14.57 = 1 + 0.1635 = 1.1635$.

• PROBLEM 10-31

Find log 2.563.

Solution: From the table,

$\log 2.57 = 0.4099$,

$\log 2.56 = 0.4082$

Hence we can use the following chart to find log 2.563

Number	Log of the Number
2.5700	.4099
2.5630	?
2.5600	.4082

.010 .003 .0017 x

The difference between 2.57 and 2.56 is .0100

The difference between log 2.57 and log 2.56 is .0017.

The difference between 2.560 and 2.563 is .0030.

and the difference between log 2.560 and log 2.563 is x.

We proceed with the following ratio:

$$\frac{\text{Difference between } 2.563 + 2.560}{\text{Difference between } 2.570 + 2.560}$$

$$= \frac{\text{Difference between } \log 2.563 + \log 2.560}{\text{Difference between } \log 2.570 + \log 2.560}$$

Replacing these differences with our above data we obtain the following ratio:

$$\frac{.0030}{.0100} = \frac{x}{.0017}$$

Cross multiply: $(.0100)x = (.0030)(.0017)$

$$.0100\,x = .0000051$$

Multiplying each side by 100 (moving each decimal point 2 digits to the right: $x = .00051$

Thus the difference between log 2.560 and log 2.563 is .00051, or rounded off to the nearest ten thousandth is .0005. Therefore

$$\log 2.560 + .0005 = \log 2.563$$

$$.4082 + .0005 = \log 2.563$$

$$\log 2.563 = 0.4087$$

• PROBLEM 10-32

Find the logarithm of 0.003124.

Solution: $0.003124 = 3.124 \times 10^{-3}$ and -3, the power of 10, is the characteristic. From a table of logarithms, .4942 and .4955 are the mantissas for 3.12 and 3.130 respectively. The mantissa for 3.124 can be found by interpolation.

$$.01\left[\begin{array}{l} .004\left[\begin{array}{l}3.120\\3.124\end{array}\right.\\ \quad 3.130\end{array}\right. \qquad \begin{array}{c}\log N\\ \left.\begin{array}{r}.4942\\x\end{array}\right]d \\ .4955\end{array}\Bigg].0013$$

Setting up the following proportion:

$\dfrac{.004}{.01} = \dfrac{d}{.0013}$. Cross multiply to obtain $(.01)d = (.004)(.0013)$

$$d = (.0013)\left(\frac{.004}{.01}\right) = \frac{(1.3 \times 10^{-3})(4 \times 10^{-3})}{1 \times 10^{-2}}$$

$$= \frac{5.2 \times 10^{-6}}{1 \times 10^{-2}}$$

$$= 5.2 \times 10^{[-6-(-2)]} = 5.2 \times 10^{-4}$$

$$= (5.2)(0.0001)$$

$$d = 0.00052$$

Therefore, $x = \log 3.124 = 0.4942 + 0.00052 = 0.49472$. Hence, the mantissa for 3.124 is 0.49472. Therefore, $\log 0.003124 = -3 + 0.49472$

$= -2.50528$

or

$$7.49472 - 10 .$$

• PROBLEM 10-33

Find the logarithm of 3614.0.

Solution: $3,614 = 3.614 \times 10^3$, hence our characteristic (exponent of 10) is 3. To determine the mantissa of 3.614, since our log tables only give us values for 3.61 and 3.62, we make the following interpolation:

		N	log N		
0.01	0.004	3.61	0.5575	d	.0012
		3.614	x		
		3.62	0.5587		

The following proportion is set up:

$$\frac{0.004}{0.01} = \frac{d}{0.0012}, \text{ cross multiplying we obtain } (.01)d = (.004)(.0012)$$

or $$d = \frac{0.004}{0.01}(0.0012) = \frac{(4 \times 10^{-3})(1.2 \times 10^{-3})}{1 \times 10^{-2}} = \frac{4.8 \times 10^{-6}}{1 \times 10^{-2}}$$

$$= 4.8 \times 10^{-6-(-2)} = (4.8) \times 10^{-4} = (4.8)(0.0001)$$

$$d = 0.00048$$

Hence, $x = \log 3.614 = 0.5575 + d$
$= 0.5575 + 0.00048$
$= 0.5580$

Therefore, the mantissa for 3614 is 0.5580. The characteristic is 3.

Hence, the logarithm of 3614.0 is $3 + .5580 = 3.5580$.

• PROBLEM 10-34

If $\log N = 8.35721 - 10$, find N.

Solution: The characteristic is - 2 (i.e. 8 - 10) and the mantissa 35721 is located between 3560 and 3579. These mantissas correspond to the numbers 227 and 228, respectively. Now a proportion can be set up of the form:

		N	log N (mantissa)		
1		227	35600		190
	x	?	35721	69	
		228	35790		

$$\frac{x}{1} = \frac{69}{190} \quad \text{or} \quad x = \frac{69}{190} = .36315 \approx .3632$$

Now subtract x from 228 to obtain:

$$\begin{array}{r} 228.0000 \\ \underline{.3632} \\ 227.6368 \end{array}$$

The characteristic tells us that the number must be less than one and that there must be one zero between the decimal point and the first nonzero number. (The characteristic is one more than the number of zeros between the decimal point and the first nonzero digit).

Therefore the number N is: $.02276368 \approx .022764$

• PROBLEM 10-35

Find N if $\log N = 3.6129$.

Solution: As a first step, we write this number in standard form:

$$\log N = .6129 + 3.$$

Use a table of common logarithms of numbers to find the number whose mantissa is 6129. We must use interpolation to do this.

		Number	Mantissa		
10	x	4110	6138		10
		?	6129	1	
		4100	6128		

$$\frac{x}{10} = \frac{1}{10}$$

$$x = 1$$

The number is $4100 + 1 = 4101$. Since $\log N = .6129 + 3$, the characteristic of the number is $+3$. Recall that for a number > 1 the characteristic is positive and is one less than the number of digits before the decimal point. Thus, there are four digits before the decimal point, and the desired number is 4,101.

• PROBLEM 10-36

Find N if $\log N = 0.8045 - 2$.

Solution: Note that the characteristic is -2. Therefore, the number that corresponds to the mantissa 0.8045 will be multiplied by 10^{-2}. Since the mantissa 0.8045 does not appear in a table of four place logarithms, the number that corresponds to this mantissa can be found by interpolation.

		x	log x		
.01	d	6.37	0.8041	.0004	.0007
		y	0.8045		
		6.38	0.8048		

Set up the following proportion:

$$\frac{d}{.01} = \frac{.0004}{.0007}$$

Cross multiply to obtain

$$d = (.01)\left(\frac{.0004}{.0007}\right) = \left(1 \times 10^{-2}\right)\left(\frac{4 \times 10^{-4}}{7 \times 10^{-4}}\right)$$

$$= \frac{4 \times 10^{-6}}{7 \times 10^{-4}} = \frac{4}{7} \times 10^{-2}$$

$$= \left(\frac{\cancel{4}^{1}}{7}\right)\left(\frac{1}{\cancel{100}_{25}}\right)$$

$$d = \frac{1}{175} = 0.0057 \; .$$

Thus, $y = 6.37 + d = 6.37 + .0057 = 6.3757$ and

$N = (6.3757) \times 10^{-2}$

$= 6.376 \times 10^{-2}$ rounding off to 4 significant figures

$N = 0.06376$

• PROBLEM 10-37

Find $\text{Antilog}_{10}\, 1.4850$.

Solution: By definition, $\text{Antilog}_{10}\, a = N$ is equivalent to $\log_{10} N = a$. Let $\text{Antilog}_{10} 1.4850 = N$. Hence, $\text{Antilog}_{10} 1.4850 = N$ is equivalent to $\log_{10} N = 1.4850$. The characteristic is 1. The mantissa is 0.4850.

Therefore, the number that corresponds to this mantissa will be multiplied by 10^1 or 10. The mantissas which appear in a table of common logarithms and are closest to the mantissa 0.4850 are 0.4843 and 0.4857. The number that corresponds to the mantissa 0.4850 will be found by interpolation.

		Number	Logarithms		
.01	d	3.05	0.4843	.0007	.0014
		x	0.4850		
		3.06	0.4857		

Set up the following proportion.

$$\frac{d}{.01} = \frac{.0007}{.0014}$$

cross-multiplying, $.0014d = (.01)(.0007)$, or $d = .01\left(\frac{.0007}{.0014}\right)$

$$= \left(1 \times 10^{-2}\right)\left(\frac{7 \times 10^{-4}}{1.4 \times 10^{-3}}\right)$$

$$= \frac{7 \times 10^{-6}}{1.4 \times 10^{-3}} = \frac{7}{1.4} \times \frac{10^{-6}}{10^{-3}} = 5 \times 10^{-6-(-3)}$$

$$= 5 \times 10^{-3} = 5 \times .001 = .005$$

Hence, $d = 0.005$

$x = d + 3.05$

$= 0.005 + 3.050$

$= 3.055$

Hence, $N = \text{Antilog}_{10} 1.4850 = 3.055 \times 10$

$= 30.550$

$= 30.55$

Therefore $\text{Antilog}_{10} 1.4850 = 30.55$.

• PROBLEM 10-38

Find Antilog 2.3625.

Solution: By definition, $b = \text{Antilog } a$, is equivalent to $\log b = a$. Let $N = \text{Antilog } 2.3625$. Therefore, $\log N = 2.3625$. The characteristic is 2. Hence, the number that corresponds to the mantissa 0.3625 will be multiplied by 10^2 or 100. In a table of four-place common logarithms,

the mantissas 0.3617 and 0.3636 are those given that are closest to 0.3625. Therefore, since the mantissa 0.3625 does not appear in the table, the number that corresponds to this mantissa will be found through interpolation.

		Number	Log		
.01	d	2.30	0.3617	.0008	.0019
		x	0.3625		
		2.31	0.3636		

The following proportion is now established:

$$\frac{d}{.01} = \frac{.0008}{.0019}$$

Cross multiplying,

$$.0019d = (.01)(.0008)$$

$$d = .01\left(\frac{.0008}{.0019}\right)$$

$$= 1 \times 10^{-2}\left(\frac{8 \times 10^{-4}}{1.9 \times 10^{-3}}\right)$$

$$= \frac{8 \times 10^{-2+(-4)}}{1.9 \times 10^{-3}} = \frac{8 \times 10^{-6}}{1.9 \times 10^{-3}}$$

$$= \frac{8}{1.9} \times \frac{10^{-6}}{10^{-3}}$$

$$= 4.2 \times 10^{-6-(-3)}$$

$$= 4.2 \times 10^{-3}$$

$$d = .0042$$

Hence, $x = 2.30 + 0.0042 = 2.3042 \approx 2.304$. Therefore, **Antilog** 2.3625 =

$$N = 2.304 \times 10^2$$

$$= 2.304 \times 100$$

$$= 230.4$$

• PROBLEM 10-39

Determine the value of x such that $10^x = 3.142$.

Solution: The statement $10^x = 3.142$ is equivalent by definition to $\log_{10}3.142 = x$. Thus we must find log 3.142, using the following interpolation:

		Number	Log		
.01	.002	3.140	.4969	x	.0014
		3.142			
		3.150	.4983		

We set up the proportion,

$$\frac{.002}{.01} = \frac{x}{.0014}$$

Cross multiply to obtain,

$$.01x = .0000028$$

$$x = .00028$$

$$x \approx .0003$$

Thus $\log 3.142 = .4969 + .0003 = 0.4972$

Therefore $x = \log_{10}3.142 = 0.4972$

• PROBLEM 10-40

Find the value of $(2.154)^5$.

Solution: We will use logs, in solving the given problem.

Let $x = (2.154)^5$. Now take the log of both sides:

$\log x = \log(2.154)^5$; and now by the rule $\log a^b = b \log a$ we obtain

$$\log x = 5 \log(2.154).$$

Log(2.154) is now found by interpolation, using a table of common logs. Notice 2.154 occurs between 2.150 and 2.160 which have recorded logs.

.010		N = 2.150	log x = .3324		.0021
	.006	N = 2.154	?	d	
		N = 2.160	log x = .3345		

Now set up the proportion and use scientific notation.

$$\frac{.006}{.010} = \frac{d}{.0021}$$

$$\frac{6 \times 10^{-3}}{1 \times 10^{-2}} = \frac{d}{2.1 \times 10^{-3}}$$

$$(2.1 \times 10^{-3})\left(\frac{6 \times 10^{-3}}{1 \times 10^{-2}}\right) = \left(\cancel{2.1 \times 10^{-3}}\right)\left(\frac{d}{\cancel{2.1 \times 10^{-3}}}\right)$$

$$\frac{12.6 \times 10^{-6}}{1 \times 10^{-2}} = d$$

$$12.6 \times 10^{-4} = d$$

$$(12.6)(0.0001) = d$$

$$0.00126 = d$$

or

$$d \approx 0.0013$$

Hence, $\log 2.154 = .3345 - d = .3345 - .0013$

$$\log 2.154 = .3332$$

Therefore,

$$\log x = 5 \log(2.154)$$
$$= 5(.3332)$$
$$\log x = 1.6660$$

The characteristic is 1. Therefore, the number that corresponds to the mantissa 0.6660 will be multiplied by 10^1 or 10. Using a table of common logarithms, the number that approximately corresponds to the mantissa is 4.63. Then,

$$x = (4.63)10$$

or $$(2.154)^5 = x = 46.3$$

Hence, $(2.154)^5 = 46.3$

• PROBLEM 10-41

Find $\sqrt[4]{36.91}$

Solution: We can use the rule for the log of a number raised to a power to find the solution.

$$\log a^b = b \log a$$

or $$\log (36.91)^{\frac{1}{4}} = (\tfrac{1}{4}) \log 36.91$$

log 36.91 lies between log 36.90 and log 37.00 Therefore, we set up the proportion:

10	36.90	5670		12
	36.91	?	x	
	37.00	5682		

$$\frac{9}{10} = \frac{x}{12} \text{ or cross multiply to obtain}$$

$$(.9)(12) = x, \qquad x = 10.8$$

Now subtract 10.8 from 5682

$$\begin{array}{r} 5682 \\ -\ \ 10.8 \\ \hline \end{array}$$

5671.2. Since the mantissa must be less than one, move the decimal pint four places to the left i.e. (.56712). Now that the mantissa has been determined the characteristic is determined by examining the number of decimal places to the left of the decimal point of the original number. The characteristic is one less than this number. In this case the characteristic is 1. This process is used for characteristic determination because the number is greater than one.

$$(\tfrac{1}{4}) \log 36.91 = \tfrac{1}{4}(1.56712) = .39178$$

Now the answer is obtained by finding the number whose log is .39178. Again consult the table of common logarithms.

180		.39090 = log 2.460		10
	92	.39178	x	
		.39270 = log 2.470		

Now set up a proportion.

$$\frac{92}{180} = \frac{23}{45} = \frac{x}{10} \text{ or cross multiply to obtain}$$

$$45x = 230 \quad \text{or} \quad x = \frac{230}{45},$$

$$x = 5.1$$

Now subtract 5.1 from 2470 to obtain

$$\begin{array}{r} 2470.0 \\ -\ \ \ 5.1 \\ \hline 2464.9 \end{array}$$

Now convert to the answer by placing the decimal point between the two and four. (Note: from the above interpolation we knew the answer must be the integer 2 plus a rational part.)

$$\sqrt[4]{36.91} = 2.4649$$

• PROBLEM 10-42

Calculate the number $\dfrac{3480 \times 1265}{.00143}$

Solution: One way to proceed is to write

$$\frac{3480 \times 1265}{.00143} = \frac{3.48 \times 10^3 \times 1.265 \times 10^3}{1.43 \times 10^{-3}} = \frac{3.48 \times 1.265 \times 10^6}{1.43 \times 10^{-3}},$$

and since $\dfrac{a^x}{a^y} = a^{x-y}$, we have

$$\frac{3.48 \times 1.265 \times 10^9}{1.43}$$

Then calculate the number

$$N = \frac{3.48 \times 1.265}{1.43},$$

where $\dfrac{3480 \times 1265}{.00143} = N \times 10^9$. Taking logarithms of both sides of the equation:

$$\log N = \log\left(\frac{3.48 \times 1.265}{1.43}\right)$$

Use the rules of logarithms,

$$\log (ab) = \log a + \log b$$

$$\log c/d = \log c - \log d$$

then

$$\log N = \log(3.48 \times 1.265) - \log 1.43$$

$$\log N = \log 3.48 + \log 1.265 - \log 1.43$$

To find log 3.48 and log 1.43 we use a table of logarithms of numbers from the table:

$$\log 3.48 = .5416$$

$$\log 1.43 = .1553$$

To obtain log 1.265 we interpolate:

$$\begin{array}{cc|l|cc|c} & & \underline{N} & \underline{\log N} & & \\ .01 & .005 & 1.26 & .1004 & & \\ & & 1.265 & ? & y & .0034 \\ & & 1.27 & .1038 & & \end{array}$$

$$\frac{.005}{.01} = \frac{y}{.0034}$$

$$\frac{.5}{1} = \frac{y}{.0034}$$

$$y = (.0034)(.5) = .00170$$

$$\log 1.265 = .1004 + .0017 = .1021$$

Therefore, substituting:

$$\log N = .5416 + .1021 - .1553 = .4884.$$

Therefore,

$$\frac{3480 \times 1265}{.00143} = 3.079 \times 10^9$$

We must now find N from log N. This is done by interpolating:

$$\begin{array}{cc|l|cc|c} & & \underline{N} & \underline{\log N} & & \\ & & 3.07 & .4871 & & \\ .01 & y & ? & .4884 & .0013 & .0015 \\ & & 3.08 & .4886 & & \end{array}$$

$$\frac{y}{.01} = \frac{.0013}{.0015}$$

$$\frac{y}{.01} = \frac{13}{15}$$

$$15y = .13$$

$$y = \frac{.13}{15} = .00866$$

$$N = 3.07 + .00866 = 3.07866 \approx 3.079$$

• PROBLEM 10-43

Find $\dfrac{0.3612}{456.53}$, using logarithms.

Solution: Let $x = \dfrac{0.3612}{456.53}$

We solve for x by taking the logarithm of both sides of this equation. Hence

$$\log x = \log\left[\frac{0.3612}{456.53}\right] = \log .3612 - \log 456.53,$$

from the rule $\log a/b = \log a - \log b$.

We find these logarithms by interpolation.

$$\begin{array}{cc|l|cc|c} & & \underline{x} & \underline{\log x} & & \\ & .0002 & .361 & 9.5575 - 10 & & \\ .001 & & .3612 & ? & y & .0012 \\ & & .362 & 9.5587 - 10 & & \end{array}$$

Interpolation means taking the proportions indicated above.

$$\frac{.0002}{.001} = \frac{y}{.0012}$$

We can simplify the left side by multiplying numerator and

$$5y = .0000007$$

$$y = .00000014$$

Therefore, $x = .000791 + .00000014$

$$x = .00079114$$

• PROBLEM 10-44

Find $\text{Antilog}_{10} 0.5579 - 1.$

Solution: By definition, $\text{Antilog}_{10} a = N$ is equivalent to $\log_{10} N = a$. Let $N = \text{Antilog}_{10} 0.5579 - 1$. Therefore, $\text{Antilog}_{10} 0.5579 - 1 = N$ is equivalent to $\log_{10} N = 0.5579 - 1$. The characteristic is -1. Therefore, the number that corresponds to the mantissa 0.5579 will be multiplied by 10^{-1}. The number that corresponds to the mantissa 0.5579 must be found through interpolation since only the mantissas 0.5575 and 0.5587 appear in a table of four-place logarithms, base 10.

		Number	log		
.01	d	3.61	0.5575	.0004	.0012
		x	0.5579		
		3.62	0.5587		

Set up the following proportion.

$$\frac{d}{.01} = \frac{.0004}{.0012}$$

$$d = .01\left(\frac{.0004}{.0012}\right)$$

$$= (1 \times 10^{-2})\left(\frac{4 \times 10^{-4}}{1.2 \times 10^{-3}}\right)$$

$$= \frac{4 \times 10^{-6}}{1.2 \times 10^{-3}}$$

$$= 3.3 \times 10^{-3}$$

$$= 3.3(.001)$$

$$d = 0.0033$$

Therefore,

$$x = 3.6100 + d$$
$$= 3.6100 + 0.0033$$
$$= 3.6133$$

Hence, $\text{Antilog}_{10} 0.5579 - 1 = N = 10^{-1}(3.6133)$

$$= 0.36133.$$

$$N \approx 0.3613.$$

• PROBLEM 10-45

Find the product 5.06 X 71.32 by using logs and antilogs.

Solution: By definition, antilog $a = N$ is equivalent to $\log N = a$. Now substitute the value for a in the antilog expression. Therefore, antilog $a = N$ becomes antilog$(\log N) = N$. Let $N = 5.06 \times 71.32$; then, $5.06 \times 71.32 = \text{antilog}[\log(5.06 \times 71.32)]$. Since $5.06 \times 71.32 = 5.06 \times 7.132 \times 10$, we write:

$$5.06 \times 71.32 = \text{antilog}[\log(5.06 \times 7.132 \times 10)] \qquad (1)$$

Evaluating the expression in the brackets:

$$\log(5.06 \times 7.132 \times 10) = \log 5.06 + \log 7.132 + \log 10.$$

This is true because of the following law of exponents:

$$\log abc = \log a + \log b + \log c.$$

Using a table of common logarithms to find the value of log 5.06 and noting that $\log 10 = 1$,

$$\log(5.06 \times 7.132 \times 10) = 0.7042 + (\log 7.132) + 1$$
$$= 1.7042 + \log 7.132 \qquad (2)$$

We now evaluate log 7.132. The numbers that appear in a table of common logarithms which are closest to the number 7.132 are 7.13 and 7.14. The mantissa that corresponds to the number 7.132 will be found by interpolation.

		Number	Logarithm		
.01	.002	7.13	0.8531	d	.0006
		7.132	x		
		7.14	0.8537		

Now, setting up the following proportion:

$$\frac{d}{.0006} = \frac{.002}{.01}$$

Cross-multiplying,

$$d = .0006\left(\frac{.002}{.01}\right)$$

$$= (6 \times 10^{-4})\left(\frac{2 \times 10^{-3}}{1 \times 10^{-2}}\right) = \frac{12 \times 10^{-4+(-3)}}{1 \times 10^{-2}}$$

$$= \frac{12 \times 10^{-7}}{1 \times 10^{-2}} = 12 \times 10^{-7-(-2)}$$

$$= 12 \times 10^{-5}$$

$$= (1.2 \times 10^{1}) \times 10^{-5} = 1.2 \times 10^{1+(-5)}$$

$$= 1.2 \times 10^{-4}$$

$$= 1.2 \times 0.0001$$

$$= 0.00012$$

$$\approx 0.0001$$

Hence, $\log 7.132 = x = 0.8531 + 0.0001$

$$= 0.8532.$$

Therefore, equation (2) becomes:

$$\log(5.06 \times 7.132 \times 10) = 1.7042 + 0.8532$$
$$= 2.5574$$

Equation (1) becomes:

$$5.06 \times 71.32 = \text{antilog}[2.5574] = M \qquad (3)$$

By definition, antilog[2.5574] is equivalent to $\log M = 2.5574$. The characteristic is 2. The mantissa is 0.5574. The number that corresponds to this mantissa will be multiplied to 10^2 or 100. The mantissas which appear in a table of logarithms and are closest to the mantissa 0.5574 are 0.5563 and 0.5575. The number that corresponds to the mantissa 0.5574 can be found by interpolation.

		Number	Logarithm		
.01	d	3.60	0.5563	.0011	.0012
		x	0.5574		
		3.61	0.5575		

Now setting up the following proportion:

$$\frac{d}{.01} = \frac{.0011}{.0012}$$

Cross-multiplying, $d = .01\left(\frac{.0011}{.0012}\right)$

$$= \left(1 \times 10^{-2}\right)\left(\frac{1.1 \times 10^{-3}}{1.2 \times 10^{-3}}\right) = \frac{1.1 \times 10^{-2+(-3)}}{1.2 \times 10^{-3}}$$

$$= \frac{1.1 \times 10^{-5}}{1.2 \times 10^{-3}} = \frac{1.1}{1.2} \times 10^{-5-(-3)}$$

$$= 0.917 \times 10^{-2}$$

$$= 0.917 \times 0.01$$

$$= 0.00917$$

$$\approx 0.009$$

Hence, $x = 3.60 + 0.009$

$$= 3.600 + 0.009$$

$$= 3.609$$

Therefore, $M = 3.609 \times 10^2 = 360.9$. Equation (3) becomes:

$$5.06 \times 71.32 = M = 360.9$$

• PROBLEM 10-46

Calculate $\frac{50.73}{2.42}$, using logs and antilogs.

Solution: By definition, antilog a = N is equivalent to log N = a. Now substitute the value for a in the antilog expression. Therefore, antilog a = N becomes antilog(log N) = N. Let

$$N = \frac{50.73}{2.42} ;$$

then

$$\frac{50.73}{2.42} = \text{antilog}\left[\log\frac{50.73}{2.42}\right].$$

Since

$$\frac{50.73}{2.42} = \frac{5.073 \times 10}{2.42} ,$$

we write:

$$\frac{50.73}{2.42} = \text{antilog}\left[\log\frac{5.073 \times 10}{2.42}\right] \qquad (1)$$

Evaluating the expression in the brackets:

$$\log\frac{5.073 \times 10}{2.42} = (\log 5.073 + \log 10) - \log 2.42.$$

This statement is true by the two following laws of exponents:

$$\log ab = \log a + \log b$$

$$\log\frac{c}{d} = \log c - \log d$$

Noting that $\log 10 = 1$, and using a table of common logarithms to obtain $\log 2.42$, we obtain:

$$\log\frac{5.073 \times 10}{2.42} = (\log 5.073 + 1) - 0.3838,$$

or

$$\log\frac{5.073 \times 10}{2.42} = (\log 5.073) + 1 - 0.3838 \qquad (2)$$

Now, to find log 5.073, we use interpolation.

		Number	Logarithm		
.01	.003	5.07	0.7050	d	.0009
		5.073	x		
		5.08	0.7059		

Set up the following proportion:

$$\frac{d}{.0009} = \frac{.003}{.01}$$

Cross-multiplying, $d = .0009\left(\frac{.003}{.01}\right)$

$$= \left(9 \times 10^{-4}\right)\left(\frac{3 \times 10^{-3}}{1 \times 10^{-2}}\right) = \frac{27 \times 10^{-4+(-3)}}{1 \times 10^{-2}}$$

$$= \frac{27 \times 10^{-7}}{1 \times 10^{-2}} = \frac{27}{1} \times 10^{-7-(-2)} = 27 \times 10^{-7+2}$$

$$= 27 \times 10^{-5}$$

$$= 2.7 \times 10^{1} \times 10^{-5}$$

$$= 2.7 \times 10^{-4}$$

$$d = 0.00027 \approx 0.0003$$

Hence, $x = 0.7050 + d = 0.7050 + 0.0003$. Therefore,

$$\log 5.073 = x = 0.7053.$$

Rewriting equation (2):

$$\log\frac{5.073 \times 10}{2.42} = 0.7053 + 1 - 0.3838$$

$$= 1.7053 - 0.3838$$

$$= 1.3215$$

Therefore, equation (1) becomes:

$$\frac{50.73}{2.42} = \text{antilog}[1.3215] = M \qquad (3)$$

By definition, antilog[1.3215] is equivalent to log M = 1.3215. The characteristic is 1. The mantissa is 0.3215. The number that corresponds to this mantissa will be multiplied by 10^1 or 10. The mantissas which appear in a log table and are closest to the mantissa 0.3215 are 0.3201 and 0.3222. The number that corresponds to the mantissa 0.3215 will be found by interpolation.

	Number	Logarithm		
.01	d [2.09	0.3201	.0014	0.0021
	x	0.3215		
	2.10	0.3222		

Now, set up the following proportion.

$$\frac{d}{.01} = \frac{.0014}{.0021}$$

Cross-multiplying , $d = (.01)\left(\frac{.0014}{.0021}\right)$

$$= \left(1 \times 10^{-2}\right)\left(\frac{1.4 \times 10^{-3}}{2.1 \times 10^{-3}}\right) = \frac{1.4 \times 10^{-2-3}}{2.1 \times 10^{-3}}$$

$$= \frac{1.4 \times 10^{-5}}{2.1 \times 10^{-3}} = \frac{1.4}{2.1} \times 10^{-5-(-3)}$$

$$= 0.67 \times 10^{-2}$$

$$= 0.67 \times 0.01$$

$$d = 0.0067 \approx 0.007$$

Hence, $x = 2.09 + d$

$$= 2.09 + 0.007$$

$$= 2.090 + 0.007$$

$$= 2.097$$

Therefore, $M = 2.097 \times 10 = 20.97$. Hence, equation (3) becomes:

$$\frac{50.73}{2.42} = \text{antilog}[1.3215]$$

$$= M$$

$$= 20.97 \ .$$

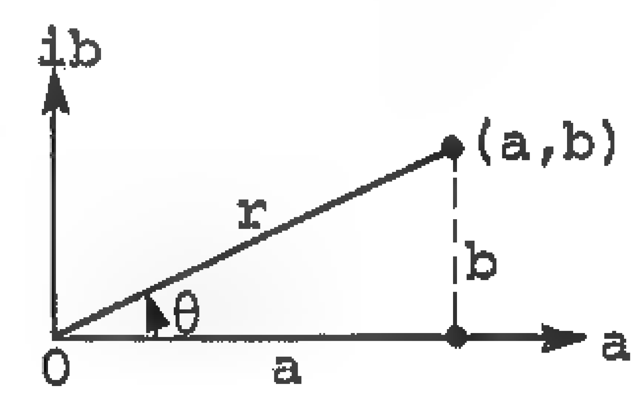
CHAPTER 11

COMPLEX NUMBERS

• PROBLEM 11-1

Find the amplitude and the modulus of $5 - 3i$.

Solution: The complex number $5 - 3i$ is expressed in the form $a + ib$. Here the modulus is the length (distance) r from the origin 0 to the point (a,b) and the amplitude is the angle θ, measured clockwise, that



the distance makes with respect to the x axis.

The amplitude θ is determined by the form

$$\tan \theta = \frac{b}{a} \quad \text{or} \quad \arctan \frac{b}{a} = \theta.$$

Since $a > 0$ and $b < 0$, θ lies in the fourth quadrant, that is $270° < \theta < 360°$. From the table we find

$$\arctan \frac{3}{5} = \arctan 0.6000 = 31°0'$$

Hence, $\theta = 360° - 31° = 329°$, the amplitude. The modulus is $r = \sqrt{5^2 + (-3)^2} = \sqrt{34}$.

• PROBLEM 11-2

Express $-6 + 8i$ in trigonometric form.

Fig. A

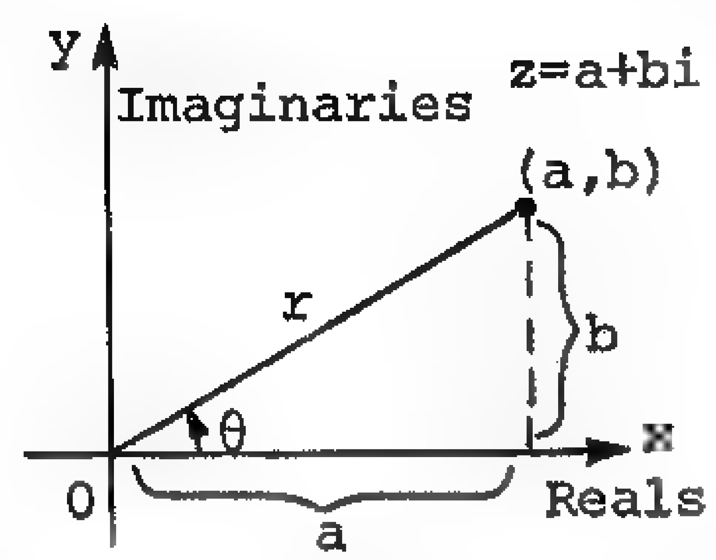
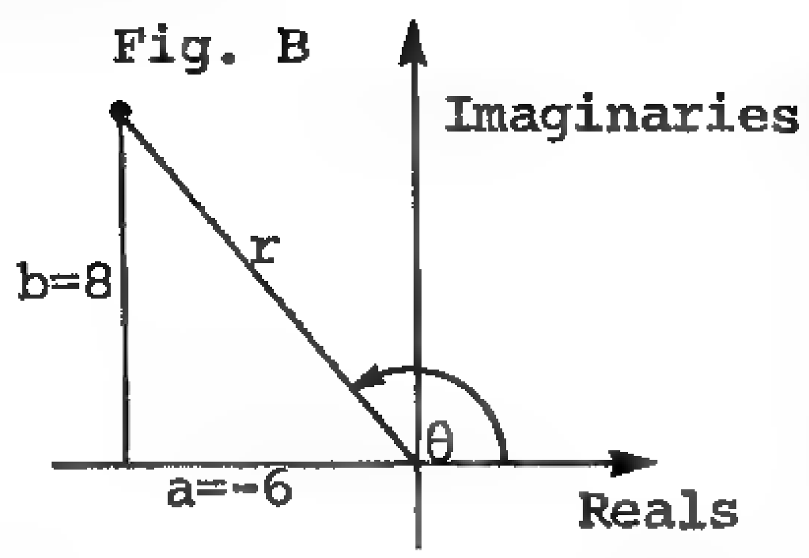


Fig. B



Solution: We are given a complex number z in the form of $a + bi$, where a and b are real numbers and (a,b) is the corresponding point in the cartesian plane. The value of a is found on the real axis and b is located on the imaginary axis. (See Figure A). Now let r denote the distance between the origin and the point which represents z and let θ be an angle in standard position whose terminal side contains the point z. We want to express a and b in terms of θ.

$$\cos\theta = \frac{a}{r} \quad \Rightarrow \quad a = r\cos\theta$$

$$\sin\theta = \frac{b}{r} \quad \Rightarrow \quad b = r\sin\theta$$

Thus, $a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

In this example, $a = -6$ and $b = 8$ (see Figure B). To find r, apply the Pythagorean theorem.

$$r^2 = a^2 + b^2 = (-6)^2 + (8)^2 = 100 ;$$

thus, $r = 10$.

For θ: $\tan\theta = \frac{b}{a} = \frac{8}{-6} = -\frac{4}{3} \approx -1.333 \ldots$. First look up in a table of trigonometric functions the reference angle whose tangent is 4/3. It is 53.1°. But θ is in Quadrant II as we note from Figure B. Hence $\theta = 180^\circ - 53.1^\circ = 126.9^\circ$. Therefore, since $a + bi = r(\cos\theta + i\sin\theta)$,

$$-6 + 8i = 10(\cos 126.9^\circ + i\sin 126.9^\circ).$$

• PROBLEM 11-3

Express each of the following in rectangular form, $a + bi$. (a) $3(\cos 30^\circ + i\sin 30^\circ)$

(b) $10(\cos 180^\circ + i\sin 180^\circ)$

Solution: The complex numbers as given are in the trigonometric form

$$r(\cos\theta + i\sin\theta)$$

in part a, $\theta = 30^\circ$,

(a) $3(\cos 30^\circ + i\sin 30^\circ) = \frac{3}{2}\sqrt{3} + \frac{3}{2}i$

in part b, $\theta = 180^\circ$,

(b) $10(\cos 180^\circ + i\sin 180^\circ) = 10(-1 + i \cdot 0) = -10$

Check: $r^2 = x^2 + y^2$

part a: $(3)^2 = \left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{27}{4} + \frac{9}{4} = 9$

part b: $(10)^2 = (-10)^2$

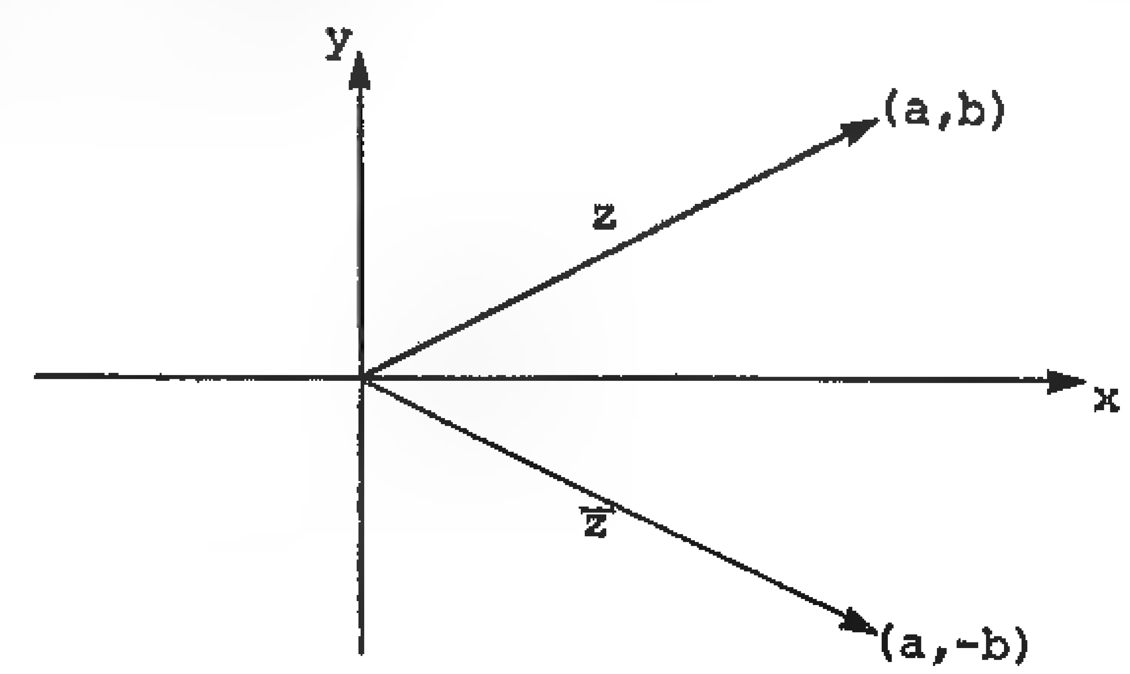
• PROBLEM 11-4

Find the four fourth roots of 16, including any imaginary roots.

Solution: The fourth roots of 16 are given by the radical $\sqrt[4]{16}$. Let $N = \sqrt[4]{16}$. The equation $N = \sqrt[4]{16}$ is equivalent to $N^4 = 16$, or $(N)(N)(N)(N) = 16$. If $N = \pm 2$, then $N^4 = (\pm 2)^4 = 16$. Hence, two of the four fourth roots of 16 are +2 and -2. Also, note that $i^4 = \left(i^2\right)^2 = (-1)^2 = 1$. Hence, if $N = \pm 2i$, then $N^4 = (\pm 2i)^4 = (\pm 2)^4 \cdot i^4 = 16(1) = 16$. Therefore, $+2i$ and $-2i$ are also two fourth roots of 16. Then, the four fourth roots of 16 are $+2, -2, +2i$, and $-2i$.

• PROBLEM 11-5

What is the conjugate of $3 - 2i$ and the conjugate of $5 + 7i$?



Solution: Any complex number may be interpreted as an ordered pair in the plane with the real component designated by the x value and the imaginary part designated by the y value. The conjugate of a complex number is that number which when multiplied by the original

complex number yields a product which is purely real. Geometrically, the complex conjugate is a reflection of the complex number through the x-axis. The complex conjugate of $3 - 2i$ is $3 + 2i$

i.e., $$(3 - 2i)(3 + 2i) = 13 .$$

The conjugate of $5 + 7i$ is $5 - 7i$

$$(5 + 7i)(5 - 7i) = 74 .$$

The conjugate of a pure real number a, which can be written $a + 0i$, is merely itself or $a - 0i$. Geometrically we see that the reflection of a real number is actually itself. The conjugate of a pure imaginary number bi is $-bi$. The conjugate of a complex number $a + bi$ is $a - bi$.

• PROBLEM 11-6

Add $(3 + 4i)$ and $(2 - 5i)$.

Solution: Numbers of the form $a + bi$, where a and b are real numbers, are called complex numbers. In the complex number $a + bi$, a is called the real part and bi is called the imaginary part. To add two complex numbers, add the real parts and add the pure imaginary parts. Therefore: we have

$$\begin{aligned}(3 + 4i) + (2 - 5i) &= (3 + 2) + (4 - 5)i \\ &= 5 + (-i) \\ &= 5 - i\end{aligned}$$

Or we may treat the problem as the sum of two binomials:

$$\begin{aligned}(3 + 4i) + (2 - 5i) &= 3 + 4i + 2 - 5i \\ &= 5 - i\end{aligned}$$

• PROBLEM 11-7

Perform the indicated operations: $(-2 + 3i) + [5 + (-6)i]$.

Solution: Addition of complex numbers is defined in the following way:

$$(a + bi) + (c + di) = (a + c) + (b + d)i ,$$

where a,b,c,d are real numbers. Thus,

$$\begin{aligned}(-2 + 3i) + [5 + (-6)i] &= (-2 + 5) + [3 + (-6)]i \\ &= 3 + (-3)i \\ &= 3 - 3i.\end{aligned}$$

• PROBLEM 11-8

Write each of the following in the form $a + bi$.

a) $(2 + 4i) + (3 + i)$

b) $(2 + i) - (4 - 2i)$

c) $(4 - i) - (6 - 2i)$

d) $3 - (4 + 2i)$

Solution:

a) $$\begin{aligned}(2 + 4i) + (3 + i) &= 2 + 4i + 3 + i \\ &= (2 + 3) + (4i + i) \\ &= 5 + 5i\end{aligned}$$

b) $$\begin{aligned}(2 + i) - (4 - 2i) &= 2 + i - 4 + 2i \\ &= (2 - 4) + (i + 2i) \\ &= -2 + 3i\end{aligned}$$

c) $$\begin{aligned}(4 - i) - (6 - 2i) &= 4 - i - 6 + 2i \\ &= (4 - 6) + (-i + 2i) \\ &= -2 + i\end{aligned}$$

d) $$\begin{aligned}3 - (4 + 2i) &= 3 - 4 - 2i \\ &= (3 - 4) - 2i \\ &= -1 - 2i\end{aligned}$$

• PROBLEM 11-9

Find the product $(2 + 3i)(-2 - 5i)$.

Solution: Using the following method: product of first elements, + product of outer elements + product of inner elements + product of last elements:

$$\begin{aligned}(2 + 3i)(-2 - 5i) &= 2(-2) + 2(-5i) + 3i(-2) + 3i(-5i) \\ &= -4 - 10i - 6i - 15i^2 \\ &= -4 - 16i - 15i^2\end{aligned}$$

Recall $i^2 = -1$, hence, $= -4 - 16i - 15(-1)$

$$= -4 - 16i + 15$$

$$= 11 - 16i$$

The same result is obtained by using the distributive law.

$$(2 + 3i)(-2 - 5i) = (2 + 3i)(-2) - (2 + 3i)5i$$

$$= -4 - 6i - 10i - 15i^2 = 11 - 16i.$$

In other words, if one multiples $2 + 3i$ and $-2 - 5i$ as if they were polynomials and replaces i^2 by -1, then the correct product is obtained.

• PROBLEM 11-10

Multiply $(3 + 4i)$ by $(5 + 2i)$.

Solution: To multiply two complex numbers, form the product treating i as an ordinary number and then replace i^2 by -1. Hence,

$$(3 + 4i)(5 + 2i) = 15 + 20i + 6i + 8i^2 = 15 + 26i + 8i^2$$
$$= 15 + 26i - 8 = 7 + 26i$$

Or we may treat the problem as the product of two binomials:

$$\begin{aligned}(3 + 4i)(5 + 2i) &= (3 + 4i)5 + (3 + 4i)2i \\ &= 15 + 20i + 6i + 8i^2 \\ &= 15 + 26i + 8(-1) \\ &= 15 + 26i - 8 \\ &= 7 + 26i\end{aligned}$$

• PROBLEM 11-11

Compute the sum and product of the complex numbers $3 + 2i$ and $1 - 3i$.

Solution:

$$(3 + 2i) + (1 - 3i) = 3 + 2i + 1 - 3i = 3+1 +2i - 3i= 4-i$$

$$(3 + 2i)(1 - 3i) = 3(1) +2i(1) + 3(-3i) + 2i(-3i)$$
$$= 3 + 2i - 9i - 6i^2$$

$$(3 + 2i)(1 - 3i) = 3 - 7i - 6i^2 \qquad (1)$$

Since $i^2 = -1$, equation (1) becomes:

$$\begin{aligned}(3 + 2i)(1 - 3i) &= 3 - 7i - 6(-1) \\ &= 3 - 7i + 6 \\ &= 9 - 7i.\end{aligned}$$

• PROBLEM 11-12

Find the values of the following expressions:

a. $(2 + 3i) + (6 - 2i)$

b. $(2 - i)(1 + 3i)$

c. $i - (2 + 3i)$

Solution: a) $(2 + 3i) + (6 - 2i) = 2 + 3i + 6 -2i$

$$= 2 + 6 + 3i - 2i$$
$$= 8 + i$$

b) $(2 - i)(1 + 3i) = 2(1) - i(1) + 2(3i) -i(3i)$

$$= 2 -i + 6i - 3i^2 \qquad (1)$$

Since $i^2 = -1$, equation (1) becomes:

$$\begin{aligned}(2 - 1)(1 + 3i) &= 2 - i + 6i - 3(-1) \\ &= 2 - i + 6i + 3 \\ &= 2 + 3 - i + 6i \\ &= 5 + 5i\end{aligned}$$

c) $i - (2 + 3i) = i - 2 - 3i$

$$= -2 + i - 3i$$
$$= -2 - 2i$$

• PROBLEM 11-13

Express each of the following complex numbers in the form $a + bi$, where a and b are real:

(a)	7	(b)	$2i$	(c)	$\sqrt{-3}$
(d)	$\frac{1 + \sqrt{-3}}{2}$	(e)	$(3 + 2i)(3 - 2i)$	(f)	$\frac{1 + i}{1 - i}$

Solution:

(a) $7 = 7 + 0i \quad (a = 7, \quad b = 0)$

(b) $2i = 0 + 2i \quad (a = 0, \quad b = 2)$

(c) $\sqrt{-3} = \sqrt{(-1)(3)} = \sqrt{-1} \cdot \sqrt{3} = i\sqrt{3} = 0 + i\sqrt{3} \quad (a = 0, b = \sqrt{3})$

(d) $\frac{1 + \sqrt{-3}}{2} = \frac{1 + i\sqrt{3}}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \left(a = \frac{1}{2}, \ b = \frac{\sqrt{3}}{2}\right)$

(e) $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2 = 9 - 4i^2 = 9 - 4(-1)$
$= 9 + 4 = 13 = 13 + 0i \qquad (a = 13, \ b = 0)$

(f) In order to evaluate $\frac{1 + i}{1 - i}$, we want to eliminate the complex expression in the denominator. Therefore, we multiply numerator and denominator by the complex conjugate of the denominator, $1 + i$:

$$\frac{1 + i}{1 - i} = \frac{1 + i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{1 + 2i + i^2}{1 - i^2} = \frac{1 + 2i + (-1)}{1 - (-1)}$$
$$= \frac{2i}{2} = i = 0 + 1i \quad (a = 0, \ b = 1).$$

• PROBLEM 11-14

Divide $(6 + 3i)$ by $(2 + 4i)$.

Solution: The division of one complex number by another is accomplished by transforming the fraction into a fraction with a real denominator. In order to do this, multiply the numerator and the denominator of the given fraction by the conjugate of the complex denominator. Note that the numbers $(a+bi)$ and $(a-bi)$ are complex conjugates. Hence, the conjugate of the denominator, $2+4i$, is $2-4i$. Then, multiplying both the numerator and the denominator:

$$\frac{6 + 3i}{2 + 4i} = \frac{6 + 3i}{2 + 4i} \cdot \frac{2 - 4i}{2 - 4i}$$
$$= \frac{(6 + 3i)(2 - 4i)}{(2 + 4i)(2 - 4i)}$$
$$= \frac{12 + 6i - 24i - 12i^2}{4 + 8i - 8i - 16i^2}$$
$$= \frac{12 - 18i - 12(-1)}{4 - 16(-1)}$$
$$= \frac{12 - 18i + 12}{4 + 16}$$
$$= \frac{24 - 18i}{20}$$
$$= \frac{2(12 - 9i)}{2(10)}$$
$$= \frac{12 - 9i}{10}$$
$$= \frac{12}{10} - \frac{9}{10}i = \frac{6}{5} - \frac{9}{10}i$$

• PROBLEM 11-15

Find $(6 - 5i) \div (3 + 4i)$.

Solution: Write $(6 - 5i) \div (3 + 4i)$ as $\frac{6 - 5i}{3 + 4i}$. We then rationalize the denominator. Rationalize the denominator by multiplying the original fraction by a fraction equivalent to unity, consisting of the complex conjugate of the denominator of the original fraction in the numerator and denominator of the new fraction. When a complex number $a \pm ib$ is multiplied by its complex conjugate $a \mp ib$ the result is a real number. i.e.

$$(a + ib)(a - ib) = a^2 - (ib)^2 = a^2 + b^2 \ \left(\text{note: } i^2 = -1\right)$$

from the rule for the difference of two squares. Note: either number is the complex conjugate of the other.

$$\frac{6 - 5i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{18 - 15i - 24i + 20i^2}{9 + 16} = \frac{-2 - 39i}{9 + 16}$$
$$= -\frac{2 + 39i}{25}.$$

Check: We check by multiplying the quotient by the divisor. The product must be the dividend.

$$\left(-\frac{2 + 39i}{25}\right)(3 + 4i) = -\frac{1}{25}(6 + 125i - 156)$$
$$= -\frac{1}{25}(125i - 150)$$
$$= 6 - 5i.$$

• PROBLEM 11-16

Find the real and imaginary parts of

$(2 + 3i) \div (3 + 4i)$

Solution: In order to divide one complex number by another, the denominator must be converted to a real number. This can be done by multiplying the numerator and the denominator by the conjugate of the denominator. The complex numbers $a + bi$ and $a - bi$ are conjugates of each other. Therefore, the conjugate of the denominator $3 + 4i$ is $3 - 4i$. Then,

$$\frac{2 + 3i}{3 + 4i} = \frac{(2 + 3i)(3 - 4i)}{(3 + 4i)(3 - 4i)}$$
$$= \frac{6 + 9i - 8i - 12i^2}{9 + \cancel{12i} - \cancel{12i} - 16i^2}$$

$$\frac{6 + i - 12(-1)}{9 - 16(-1)}, \quad \text{since } i^2 = -1$$

$$= \frac{6 + i + 12}{9 + 16}$$

$$= \frac{18 + i}{25}$$

$$= \frac{18}{25} + \frac{1}{25}i$$

Hence, the real part of the quotient is $\frac{18}{25}$ and the imaginary part is $\frac{1}{25}$ i.

• PROBLEM 11-17

Divide $2 - \sqrt{2}i$ by $2 - i$:

Solution: This problem involves dividing one complex number by another. To perform this division, the numerator and the denominator are multiplied by the conjugate of the denominator. The conjugate of the denominator 2 - i is 2 + i. Then,

$$\frac{2 - \sqrt{2}\,i}{2 - i} = \frac{(2 - \sqrt{2}\,i)(2 + i)}{(2 - i)(2 + i)}$$

$$= \frac{4 - 2\sqrt{2}\,i + 2i - \sqrt{2}\,i^2}{4 - \cancel{2i} + \cancel{2i} - i^2}$$

$$= \frac{4 + (2 - 2\sqrt{2})i - \sqrt{2}\,(-1)}{4 - (-1)}, \text{since } i^2 = -1$$

$$= \frac{4 + (2 - 2\sqrt{2})i + \sqrt{2}}{4 + 1}$$

$$= \frac{4 + \sqrt{2} + (2 - 2\sqrt{2})i}{5}$$

$$= \frac{4 + \sqrt{2}}{5} + \frac{(2 - 2\sqrt{2})}{5}\,i$$

• PROBLEM 11-18

Simplify $\frac{3 - 5i}{2 + 3i}$.

Solution: To simplify $\frac{3 - 5i}{2 + 3i}$ means to write the fraction without an imaginary number in the denominator. To achieve this, we multiply the fraction by another fraction which is equivalent to unity, (so that the value of the original fraction is unchanged) which will transform the expression in the denominator to a real number. A fraction with this property must have the complex conjugate of the expression in the denominator of the original fraction as its numerator and denominator. The complex conjugate must be chosen because of its special property that when multiplied by the original complex number the result is real.

Note: a + bi; its complex conjugate is a - bi or they can be said to be conjugates of each other. To multiply notice that (a + bi)(a - bi) is the factored form of the difference of two squares. Thus we obtain

$$(a)^2 - (bi)^2; \; i^2 = -1; \; (a)^2 - (-1)(b)^2 \text{ or } a^2 + b^2.$$

$$\frac{3 - 5i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{6 - 9i - 10i + 15i^2}{4 - 9i^2}$$

$$= \frac{6 - 19i - 15}{4 + 9}$$

$$= \frac{-9 - 19i}{13} \text{ or } \frac{-9}{13} - \frac{19}{13}i$$

Since the resulting fraction has a rational number in the denominator, we have rationalized the denominator.

• PROBLEM 11-19

Simplify: (a) $4i - 7i^3$ (b) $\frac{2 - 3i}{5i}$

Solution: (a) Factor i in the expression to obtain,

$$i\left(4 - 7i^2\right)$$

$$i^2 = -1$$

and we obtain,

$$i(4 - 7[-1]) = i(4 + 7) = 11i$$

(b) Rationalize the denominator by multiplying the original fraction by a fraction equivalent to unity which will cause the imaginary expression in the denominator of the original fraction to be eliminated.

$\frac{i}{i}$ is suitable because $\frac{i}{i} = 1$ and $i^2 = -1$.

$$\frac{2-3i}{5i} = \frac{2-3i}{5i} \cdot \frac{i}{i} = \frac{2i-3i^2}{5i^2} = \frac{2i+3}{-5}$$

• PROBLEM 11-20

Expand $(2+3i)^3$.

Solution: The identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is still valid in the case of complex numbers. Thus, replacing a by 2 and b by 3i we obtain

$$\begin{aligned}(2+3i)^3 &= 2^3 + 3 \cdot 2^2(3i) + 3 \cdot 2(3i)^2 + (3i)^3 \\ &= 8 + 3 \cdot 4 \cdot 3 \cdot i + 3 \cdot 2(3^2i^2) + 3^3i^3 \\ &= 8 + 36i + 6(9)i^2 + 27i^3 \\ &= 8 + 36i + 54i^2 + 27i^3.\end{aligned}$$

Recalling that $i^2 = -1$, since $i = \sqrt{-1}$ and $i^2 = \sqrt{-1}\sqrt{-1} = -1$; and $i^3 = i^2(i) = (-1)i = -i$, we obtain:

$$\begin{aligned}&= 8 + 36i + 54(-1) + 27(-i) \\ &= 8 + 36i - 54 - 27i \\ &= 9i - 46.\end{aligned}$$

• PROBLEM 11-21

Evaluate $x^2 - 2x + 6$ for $x = 3 + 2i$.

Solution: Substituting the given value, we get

$$\begin{aligned}x^2 - 2x + 6 &= (3+2i)^2 - 2(3+2i) + 6 \\ &= (3+2i)(3+2i) - 6 - 4i + 6\end{aligned}$$

Since $(a+b)(c+d) = ac + ad + bc + bd$

$$\begin{aligned}x^2 - 2x + 6 &= (3)(3) + 6i + 6i + (2i)(2i) - 6 - 4i + 6 \\ &= 9 + 12i + (2i)^2 - 6 - 4i + 6\end{aligned}$$

Since $(ab)^2 = a^2b^2$

$(2i)^2 = 2^2(i)^2$

By definition $i^2 = -1$, hence $(2i)^2 = 4(-1) = -4$ and $x^2 - 2x + 6$

$$= 9 + 12i - 4 - 6 - 4i + 6$$

Combine terms, $= 5 + 8i.$

• PROBLEM 11-22

Show that $\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^3 = 1.$

Solution: Factor out $\frac{1}{2}$:

$$\begin{aligned}\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^3 &= \left[\frac{1}{2}(-1 + \sqrt{3}i)\right]^3 \\ &= \left(\frac{1}{2}\right)^3(-1+\sqrt{3}i)^3 = \frac{1}{8}(-1+\sqrt{3}i)^3.\end{aligned}$$

Then we apply the identity $(a+b)^3 = (a+b)(a+b)(a+b) = \left(a^2 + 2ab + b^2\right)(a+b) = a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^2$. Let $a = -1$, $b = \sqrt{3}i$, then

$$\begin{aligned}(-1+\sqrt{3}i)^3 &= (-1)^3 + 3(-1)^2\sqrt{3}i + 3(-1)(\sqrt{3}i)^2 + (\sqrt{3}i)^3 \\ &= -1 + 3\sqrt{3}i - 9i^2 + 3\sqrt{3}i^3 \\ &= -1 + 3\sqrt{3}i + 9 - 3\sqrt{3}i = 8.\end{aligned}$$

since $i^2 = -1$ and $i^3 = i^2\ i = (-1)i = -i$. Hence

$$\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^3 = \frac{1}{8} \cdot 8 = 1.$$

• PROBLEM 11-23

Show that $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = -1.$

Solution: Factor out $\frac{1}{\sqrt{2}}$:

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = \left[\frac{1}{\sqrt{2}}(1+i)\right]^4 = \left(\frac{1}{\sqrt{2}}\right)^4(1+i)^4 = \frac{1}{4}(1+i)^4$$

Note: $\left(\frac{1}{\sqrt{2}}\right)^4 = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Now we apply the identity $(a+b)^4 =$

$$\begin{aligned}(a+b)^2(a+b)^2 &= (a+b)(a+b)(a+b)(a+b) \\ &= \left(a^2 + 2ab + b^2\right)\left(a^2 + 2ab + b^2\right).\end{aligned}$$

Then,

$$\begin{array}{r} a^2 + 2ab + b^2 \\ a^2 + 2ab + b^2 \\ \hline a^4 + 2a^3b + a^2b^2 \qquad\qquad\qquad \\ + 2a^3b + 4a^2b^2 + 2ab^3 \qquad \\ + a^2b^2 + 2ab^3 + b^4 \\ \hline \end{array}$$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Thus, $a = 1$ $b = i$.

We obtain

$$(1 + i)^4 = 1^4 + 4i + 6i^2 + 4i^3 + i^4$$

substitute: $i^2 = -1$ and $i^3 = i^2 \cdot i = (-1)i = -i$ and $i^4 = i^2 \cdot i^2 = (-1)^2 = 1$. Then,

$$(1 + i)^4 = 1 + 4i - 6 - 4i + 1 = -4.$$

Hence

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = \frac{1}{4}(-4) = -1.$$

• PROBLEM 11-24

Show that $(a + bi) + (c + di) = (c + di) + (a + bi)$.

Solution: Use the assoiative, distributive, and commutative laws. Associate the corresponding components of the complex numbers, i.e., associate the real and imaginary parts respectively.

$$\begin{aligned} (a + bi) + (c + di) &= (a + c) + (b + d)i \\ &= a + c + bi + di \\ &= a + bi + c + di \\ &= (c + di) + (a + bi) \end{aligned}$$

We would suspect that zero is still the additive identity, but zero is a real number. Recall that the real number 5

and the complex number $5 + 0i$ represent the same number. Then the additive identity should be $0 + 0i$. Let us see whether it is. Recall that adding the additive identity to a number does not change the number. Applying the definition of addition.

$$\begin{aligned} (a + bi) + (0 + 0i) &= (a + 0) + (b + 0)i \\ &= a + bi \end{aligned}$$

This verifies that $0 + 0i$ is the additive identity.

• PROBLEM 11-25

Given $f(x) = x^3 + x + 1$, evaluate $f(1 + i)$.

Solution: $f(1 + i)$ indicates that $1 + i$ should be subsituted for x.

$$\begin{aligned} f(1 + i) &= (1 + i)^3 + (1 + i) + 1 \\ &= (1 + i)(1 + i)(1 + i) + (1 + i) + 1 \\ &= \left(1 + 2i + i^2\right)(1 + i) + (1 + i) + 1 \\ &= 1 + 2i + i^2 + i + 2i^2 + i^3 + (1 + i) + 1 \\ &= 1 + 3i + 3i^2 + i^3 + (1 + i) + 1 \\ &= 1 + 3i + 3i^2 + i^3 + 1 + i + 1 \end{aligned}$$

Note that $i^2 = -1, i^3 = i^2(i) = (-1)(i) = -i$. Then

$$\begin{aligned} f(1 + i) &= 1 + 3i + 3(-1) + (-1)i + 1 + i + 1 \\ &= 3i. \end{aligned}$$

• PROBLEM 11-26

Find the real numbers a and b such that

$$(a + bi) + (2 - 3i) = 2(-2 + i).$$

Solution: Subtract $(2 - 3i)$ from both sides of the given equation.

$$\begin{aligned} (a+bi) + \cancel{(2-3i)} - \cancel{(2-3i)} &= 2(-2+i) - (2-3i) \\ a+bi &= -4 + 2i - 2 + 3i \\ &= -4 - 2 + 2i + 3i \\ a+bi &= -6 + 5i \end{aligned}$$

Therefore, $a = -6$ and $b = 5$.

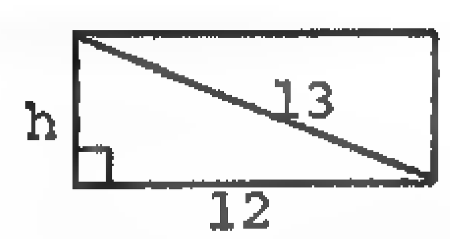
CHAPTER 12

APPLIED PROBLEMS (WORD PROBLEMS)

AREAS AND PERIMETERS

• PROBLEM 12-1

Find the area of a rectangle with base of length 12 and diagonal of length 13.



Solution: First find h, the length of the altitude of the rectangle.

In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.

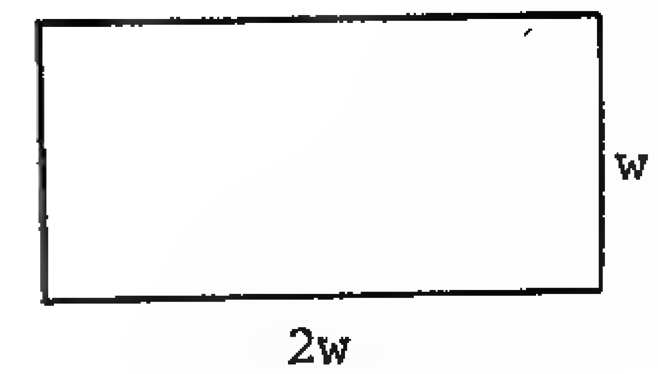
Hence, $(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2$

$$h^2 + 12^2 = 13^2$$
$$h^2 + 144 = 169$$
$$h^2 = 25,\ h = 5$$

Since the area of a rectangle is the product of the base and altitude, the area of the rectangle $= 12(5) = 60$.

• PROBLEM 12-2

One side of a rectangle is twice the length of the other side, and the perimeter is 36. Find the area of the rectangle.



Solution: Since one side of the rectangle is twice the other side, w and 2w can be used to represent the width and the length of the rectangle.

Since the perimeter = 36,

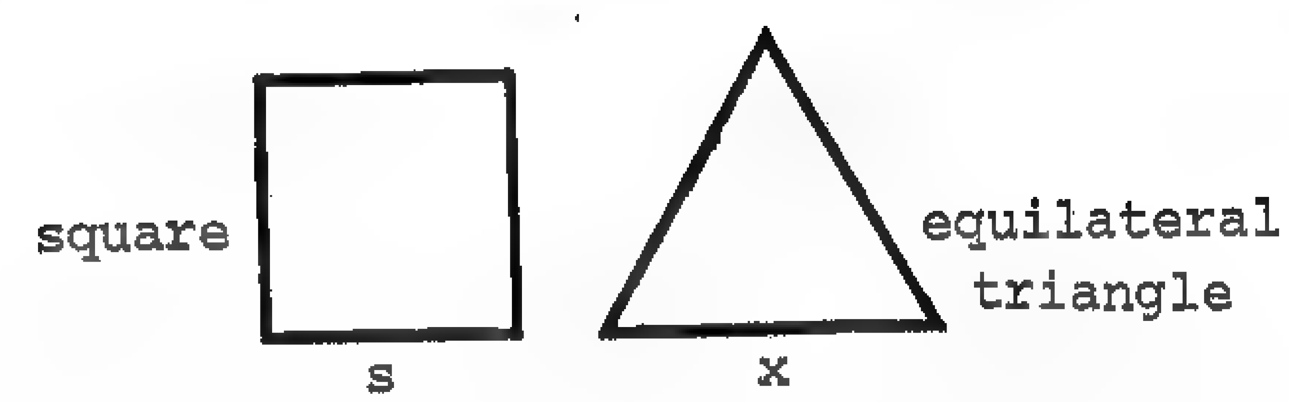
$$2w + 2w + w + w = 36$$
$$6w = 36,\ w = 6$$

Hence, the width = 6 and the length = 12.

Since the area of a rectangle is the product of the length and width, the area $= 12(6) = 72$.

• PROBLEM 12-3

The area of a square is 9. The square and an equilateral triangle have equal perimeters. Find the length of a side of the triangle.



Solution: The area of a square is the square of a side; hence, if s is the side of the square,

$$s^2 = 9$$

Take the square root of each side.

$$s = 3$$

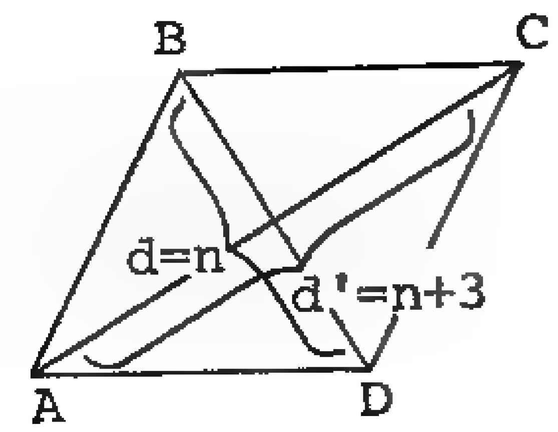
The perimeter of the square is 4 times a side. Hence, the perimeter of the square $= 4(3) = 12$.

The perimeter of the equilateral triangle is 3 times a side, that is, $3x$.

Since the perimeters are equal, $3x = 12$ and $x = 4$. Hence the length of a side of the triangle is 4.

• PROBLEM 12-4

The diagonals of a rhombus are represented by n and $n + 3$. Express the area of the rhombus in terms of n.



Solution: The area of a rhombus equals one-half the product of the diagonals.

If K = area of rhombus,

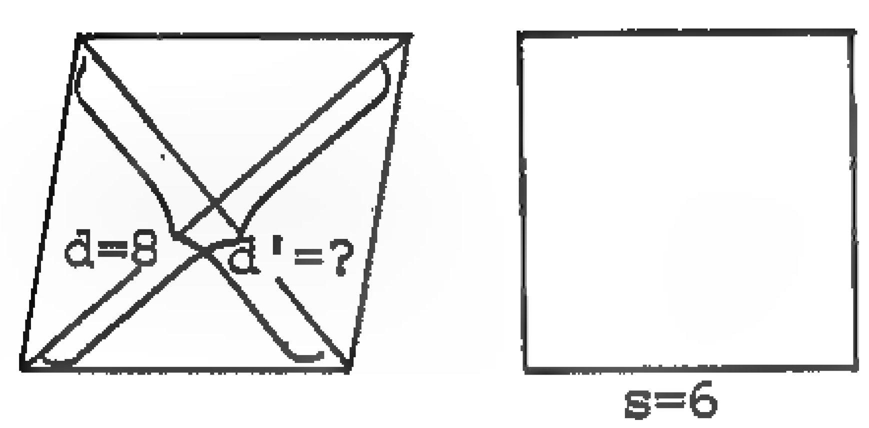
$$K = \tfrac{1}{2}dd'$$

Substitute n for d and $(n+3)$ for d'.

$$K = \tfrac{1}{2}n(n+3) \quad \text{or} \quad \tfrac{1}{2}(n^2 + 3n) \quad \text{or} \quad \frac{n^2 + 3n}{2} \;.$$

• PROBLEM 12-5

> The area of a rhombus is equal to the area of a square whose side is 6. If the length of one diagonal of the rhombus is 8, how long is the other diagonal?



Solution: Since the side, s, of the square is 6, the area of the square $= s^2 = 36$. Therefore,

$$\text{area of rhombus} = 36.$$

The area of a rhombus equals one-half the product of its diagonals.

Hence, $\tfrac{1}{2}dd' = 36$.

Substitute 8 for d.

$$\tfrac{1}{2}(8)d' = 36$$

$$4d' = 36$$

$$d' = 9$$

• PROBLEM 12-6

> A square parcel of land has twice the area of a rectangular parcel whose length is 9 feet less and whose width is 40 feet less. What are the dimensions of the square parcel of land?

Solution: Let x = the side of the square parcel of land. Then $x - 9$ is the length of the rectangular parcel, and $x - 40$ is the width of the rectangular parcel. Since the area of the square parcel is x^2, and the area of the rectangular parcel is $(x - 9)(x - 40)$, the equation relating the two areas is then

$$x^2 = 2(x - 9)(x - 40) \qquad (1)$$

$$x^2 = 2x^2 - 98x + 720 \qquad (2)$$

or $$x^2 - 98x + 720 = 0 \qquad (3)$$

In factored form, we have

$$(x - 90)(x - 8) = 0 \qquad (4)$$

So that $x = 90$ and $x = 8$.

From the statement of the problem $x = 8$ must be rejected as the side of the square parcel, because the dimensions of the rectangular parcel are 9 feet and 40 feet less than the side of the square parcel. That is, if $x = 8$ were accepted as the side of the square, then the rectangular parcel would have negative dimensions, which is impossible.

• PROBLEM 12-7

> Find the dimensions of a rectangular piece of metal whose area is 35 square inches and whose perimeter is 24 inches.

Solution: Let x = width and y = length. The area of a rectangle is equal to width times length. The perimeter of a rectangle is equal to 2 times the width plus 2 times the length. Therefore, the area may be expressed as $xy = 35$, while the perimeter can be written as $2x + 2y = 24$. Since area and perimeter are simultaneous properties of a rectangle we have the system:

$$xy = 35$$

$$2x + 2y = 24$$

The computation can be arranged in the following manner: From $xy = 35$ we obtain

$$y = \frac{35}{x}\;.$$

Using this expression for y, and substituting, $2x + 2y = 24$ becomes,

$$2x + 2\left(\frac{35}{x}\right) = 24$$

Multiplying both sides of the equation by x: (this is allowed since $x \neq 0$ because width can't $= 0$)

$$2x^2 + 70 = 24x$$

Put in standard quadratic form and solve for x by factoring:

$$2x^2 - 24x + 70 = 0$$

$$x^2 - 12x + 35 = 0$$
$$(x - 5)(x - 7) = 0$$
$$x - 5 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = 5 \quad \text{or} \quad x = 7$$

Solve for y using the area equation, $xy = 35$:

when $x = 5$, $5(y) = 35$; thus, $y = 7$

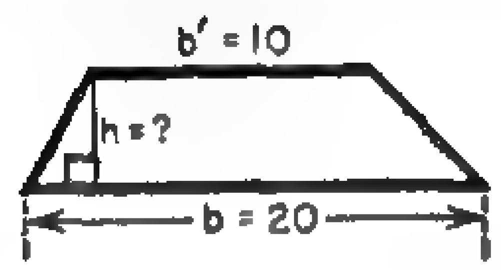
when $x = 7$, $7y = 35$; thus, $y = 5$. Then check using the perimeter equation:

$$x = 5,\ y = 7: \quad 2(5) + 2(7) = 24$$
$$x = 7,\ y = 5: \quad 2(7) + 2(5) = 24$$

Hence the solution set is $\{(5,7),(7,5)\}$. Thus the dimensions of the piece of metal are 5 by 7 inches.

• PROBLEM 12-8

The bases of a trapezoid are 10 inches and 20 inches. If the area of the trapezoid is 60 square inches, find the number of inches in the length of the altitude of the trapezoid.



Solution: Find altitude h as follows:

The area of a trapezoid equals one-half the product of the altitude and the sum of the bases.

If K = area of the trapezoid,

$$K = \frac{h}{2}(b + b')$$

Substitute 60 for K, 20 for b, and 10 for b'.

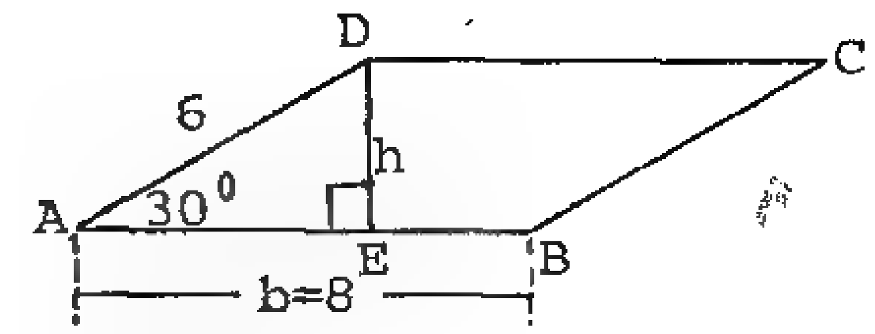
$$60 = \frac{h}{2}(20 + 10) = \frac{h}{2}(30)$$

$60 = 15h$, $h = 4$; thus the altitude is 4 inches.

• PROBLEM 12-9

In parallelogram ABCD, AB = 8 inches, AD = 6 inches, and angle $A = 30^\circ$. Find the number of square inches in the area of the parallelogram.

Solution: In right $\triangle ADE$, h is opposite 30°. Since a leg opposite 30°



equals one-half the hypotenuse,

$$h = \frac{6}{2} = 3$$

The area of a parallelogram equals the product of the base and the altitude.

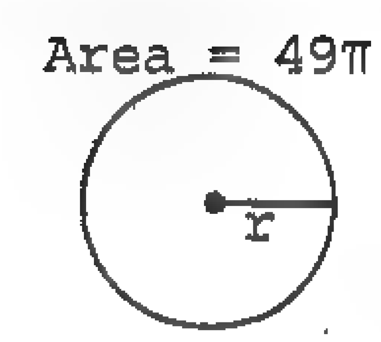
Area of ▱ABCD = bh

Substitute 8 for b and 3 for h.

Area of ▱ABCD = 8(3) = 24

• PROBLEM 12-10

The area of a circle is 49π. Find the circumference of the circle in terms of π.



Solution: First find the radius, r, as follows:

The area of a circle equals the product of π and the square of the radius.

If K = the area of the circle, then

$$K = \pi r^2$$

Substitute 49π for K.

$$49\pi = \pi r^2$$

Divide each side by π and then take the square root of each side.

$$7 = r$$

Find C, the circumference, as follows:

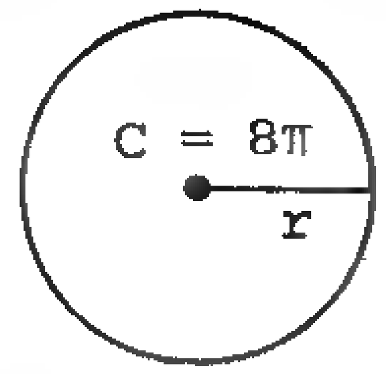
The circumference of a circle equals the product of 2π and the radius.

Hence, $C = 2\pi r$

$= 2\pi(7) = 14\pi$.

● **PROBLEM 12-11**

The circumference of a circle is 8π. What is the area of the circle in terms of π ?



Solution: First find the radius, r, as follows:

The circumference of a circle equals the product of 2π and the radius.

If C = the circumference of the circle, then

$$2\pi r = C$$

Substitute 8π for C.

$$2\pi r = 8\pi$$

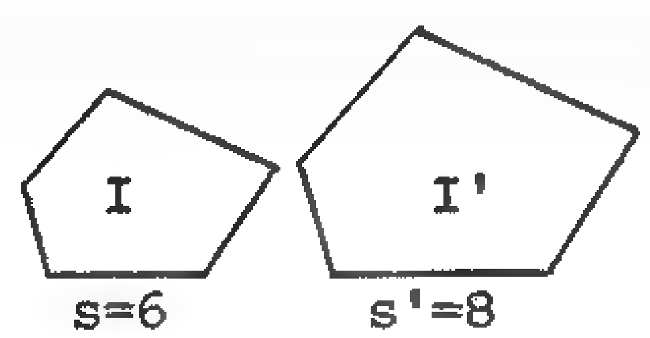
Divide each side by 2π.

$$r = 4$$

Now, since area $= \pi r^2$, area $= \pi 4^2 = 16\pi$.

● **PROBLEM 12-12**

Corresponding sides of two similar polygons are 6 and 8. If the perimeter of the smaller is 27, find the perimeter of the larger.



Solution: Find p', the perimeter of the larger polygon, by applying the following principle:

In similar polygons, the ratio of the perimeters equals the ratio of any two corresponding sides.

Hence, $\dfrac{\text{perimeter of I}'}{\text{perimeter of I}} = \dfrac{s'}{s}$.

Substitute 27 for the perimeter of I, 8 for s', and 6 for s.

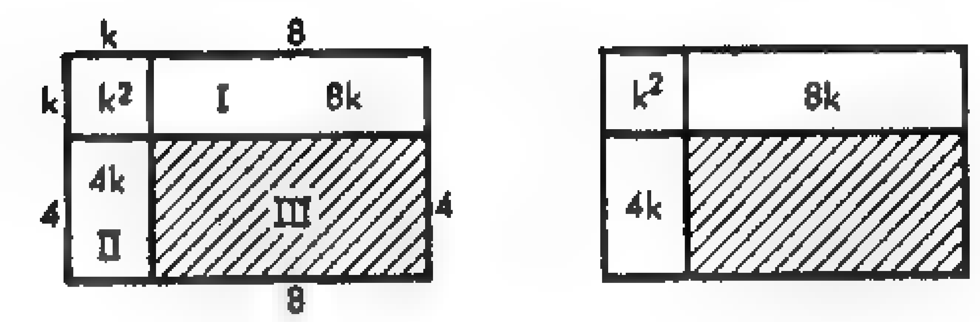
$$\frac{p'}{27} = \frac{8}{6}$$

Multiply each side by 27.

$$p' = 27\left(\frac{8}{6}\right)$$
$$= 36 \ .$$

● **PROBLEM 12-13**

In the accompanying figure, the large rectangle has been divided into a square and three smaller rectangles. If the areas of the square and two of the rectangles are k^2, $4k$, and $8k$, respectively, what is the numerical value of the area of the shaded rectangle?



Solution: Since the area of the square is k^2, the length of each side is k.

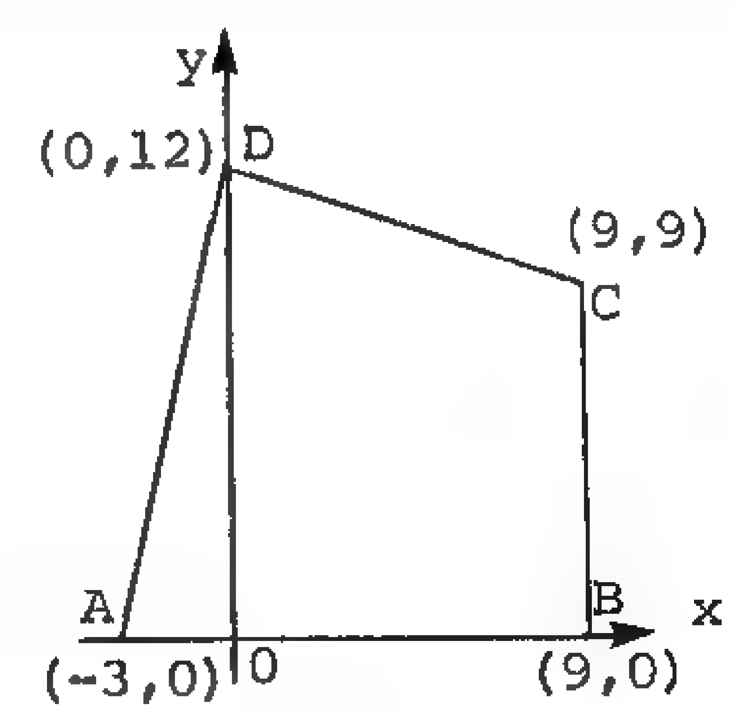
The area of rectangle I is 8k, and its altitude is k, a side of the square. Hence, its base is 8.

The area of rectangle II is 4k and its base is k, a side of the square. Hence, its altitude is 4.

Since the base and altitude of rectangle III are 8 and 4, its area is the product of 8 and 4, or 32.

● **PROBLEM 12-14**

Given quadrilateral ABCD with vertices at A(-3,0), B(9,0), C(9,9), and D(0,12). Find the area of quadrilateral ABCD.



Solution: As shown in the figure the area of quadrilateral ABCD is the sum of the areas of ΔAOD and trapezoid OBCD.

Recall that the area of a triangle is $\frac{1}{2}bh$, where b = the base and h = the altitude. In the figure note that the base of ΔAOD is 3 units and h = 12. Thus,

area of triangle = $\frac{1}{2}bh$

$= \frac{1}{2}(3)(12) = 18$

Now recall that the area of a trapezoid is $\frac{h}{2}(b + b')$, where h = altitude and b, b' = bases. In the figure note that the height of trapezoid OBCD = 9 units, and b and b' = 12 and 9. Thus,

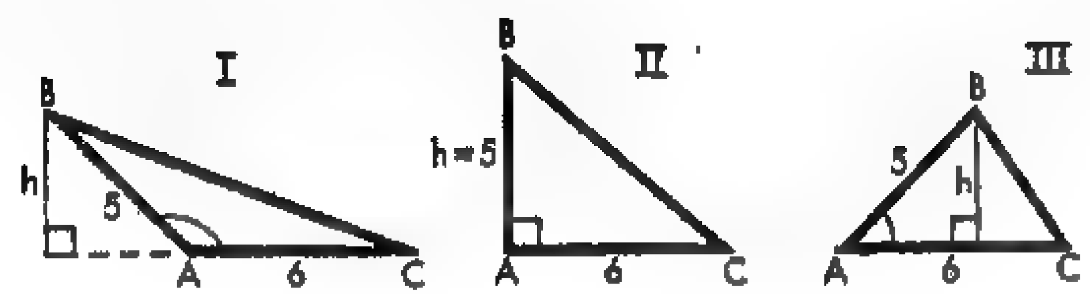
$$\text{area of trapezoid} = \frac{h}{2}(b + b')$$

$$= \frac{9}{2}(12 + 9) = \frac{9}{2}(21) = 94\frac{1}{2}$$

Area of quadrilateral ABCD = area of triangle + area of trapezoid = $18 + 94\frac{1}{2} = 112\frac{1}{2}$.

• PROBLEM 12-15

Side AB of ΔABC is 5 inches and side AC is 6 inches. If the number of degrees in angle A varies, what is the largest possible area, in square inches, of ΔABC?



Solution: If the size of $\angle A$ varies, it may be an obtuse angle, a right angle, or an acute angle. Note the three triangles in the diagram. In ΔI, $\angle A$ is an obtuse angle; in ΔII, $\angle A$ is a right angle; and in ΔIII, $\angle A$ is an acute angle. In each case, $\angle A$ is included between sides of 5 and 6 inches.

Determine the maximum area by applying the following principle:

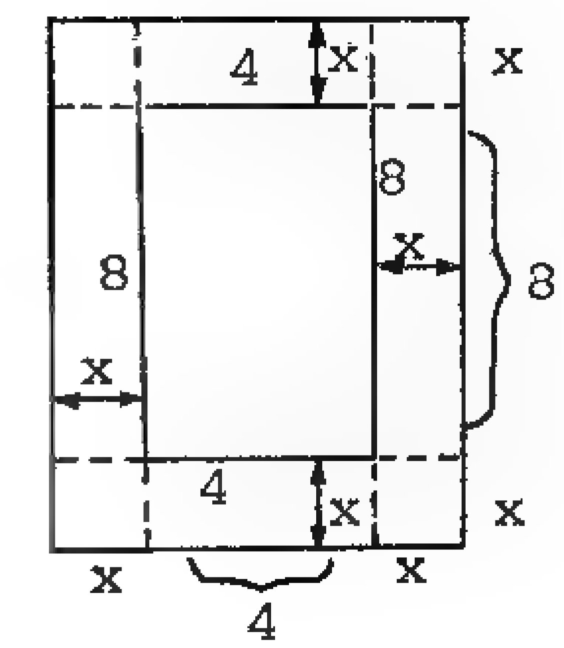
The area of a triangle equals one-half the product of a side and the altitude drawn to that side.

The largest altitude that can be drawn to side AC occurs when $\angle A$ is a right angle, as in ΔII. In ΔI and ΔIII, note that the altitude, h, is less than 5.

Hence, as $\angle A$ varies, the largest possible area is obtained when $\angle A$ is a right angle. The area of $\Delta II = \frac{1}{2}(6)(5) = 15$.

• PROBLEM 12-16

A rectangle 4 in. by 8 in. is completely bordered by a strip x in. wide. If the perimeter of the larger rectangle is twice that of the smaller rectangle, what is the value of x?



Solution: Observe the accompanying figure. The perimeter of a rectangle is the sum of the lengths of its sides. Thus, the perimeter of the larger rectangle is

$$(4+x+x) + (8+x+x) + (4+x+x) + (8+x+x) = 24 + 8x,$$

and the perimeter of the smaller or inner rectangle is $8 + 4 + 8 + 4 = 24$. We are told that the perimeter of the larger rectangle is twice that of the smaller, thus

$$24 + 8x = 2 \cdot 24$$

$$24 + 8x = 48$$

$$8x = 24$$

$$x = 3 \text{ in.}$$

• PROBLEM 12-17

A rectangle is twice as long as it is wide. If it is bordered by a strip 2 ft. wide, its area is increased by 160 sq. ft. What are its dimensions?

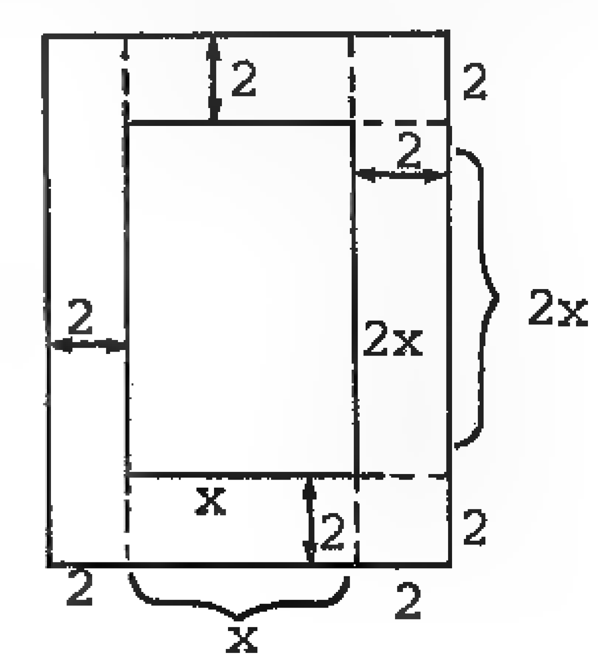
Solution: Observe the accompanying figure.

Let x = the width of the inner rectangle
Let $2x$ = the length of the inner rectangle.

The area of a rectangle is its length multiplied by its width. Thus, the area of the inner rectangle is $2x \cdot x = 2x^2$.

The length of the outer rectangle is $2 + 2x + 2 = 2x + 4$ and the width of the outer rectangle is $2 + x + 2 = x + 4$.

Thus, the area of the outer rectangle is $(2x+4)(x+4)$. We are told



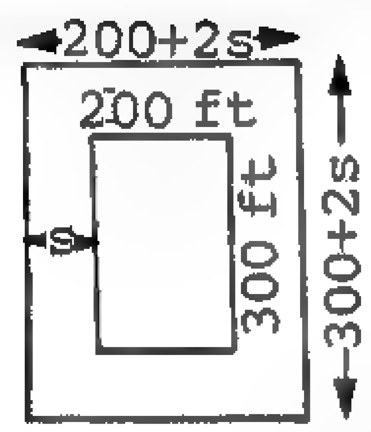
that the outer rectangle has an area of 160 sq. ft. greater than the inner, thus

$$(x+4)(2x+4) = 2x^2 + 160$$
$$2x^2 + 12x + 16 = 2x^2 + 160$$
$$12x + 16 = 160$$
$$12x = 144$$
$$x = 12, \quad 2x = 24.$$

Thus, the width of the inner rectangle is 12 ft. and the length is 24 ft.

• PROBLEM 12-18

> A supermarket, rectangular in shape and 200 feet by 300 feet, is to be built on a city block that contains 81,600 square feet. There will be a uniform strip around the building for parking. How wide is the strip?



Solution: If the strip is s feet wide, the dimensions of the supermarket will be $200 + 2s$ by $300 + 2s$ (see figure). Its area, the product of the width and length, is $(200 + 2s)(300 + 2s)$ square feet. But the area is given as 81,600 square feet. Thus, we have

$$(200 + 2s)(300 + 2s) = 81,600$$
$$60,000 + 1000s + 4s^2 = 81,600$$
$$4s^2 + 1000s - 21,600 = 0 \quad \text{Standard Quadratic Form}$$
$$s^2 + 250s - 5400 = 0 \quad \text{Dividing by 4}$$

Using the quadratic formula $s = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = 250$, and $c = -5400$, we have:

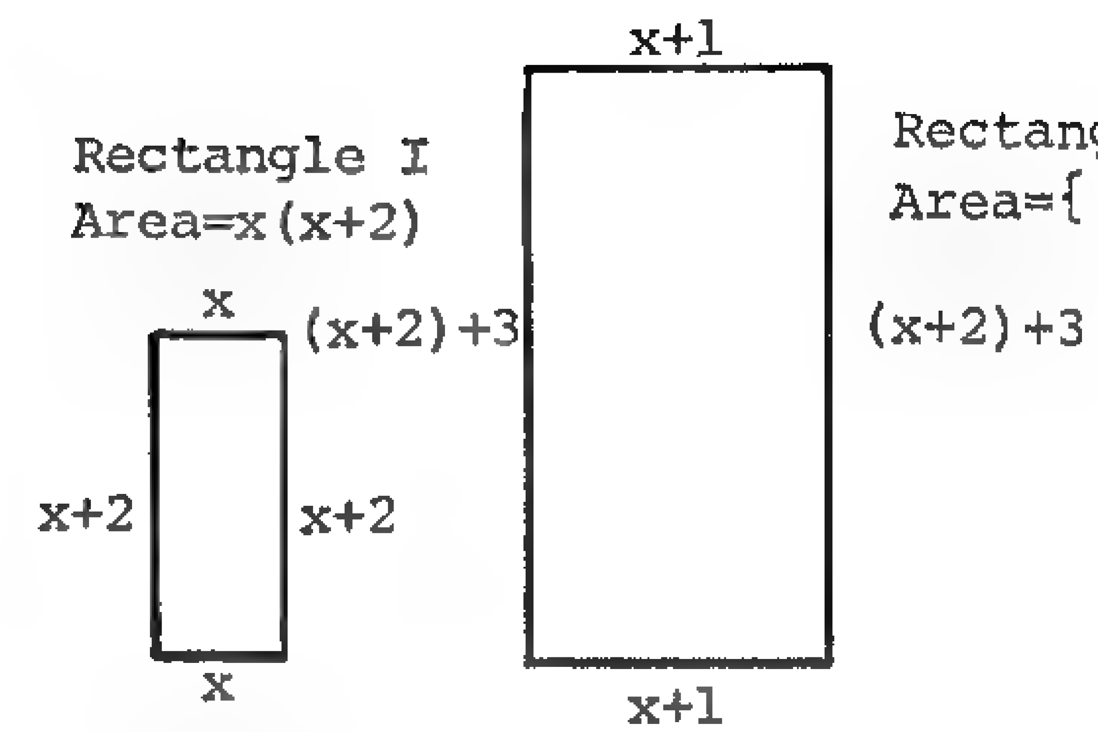
$$s = \frac{-250 \pm \sqrt{250^2 + 21,600}}{2}$$
$$s = \frac{-250 \pm 290}{2}$$
$$s = 20 \quad \text{or} \quad s = -270$$

The strip is 20 feet wide, since it is impossible for a strip to be a negative width.

Check: If the strip is 20 feet wide, then the block is 340 by 240 feet, and its area must be $(340)(240) = 81,600$ square feet.

• PROBLEM 12-19

> A rectangle has its length 2 feet greater than its width. If the length is increased by 3 feet and the width by one foot, the area of the new rectangle will be twice the area of the old. What is the length and width of the original rectangle?



Solution: We designate the original rectangle by Rectangle I, and the new rectangle by Rectangle II. Let x = the number of feet in the width of Rectangle I.

If x represents the number of feet in the width of the original rectangle, the number of feet in the length is $x + 2$. (This is obtained from the first and last sentence.) From the second sentence we gain the basic equation

$$(\text{area of new rectangle}) = 2(\text{area of original rectangle}) \qquad (1)$$

Since the area of a rectangle is the product of the width and the length the area of the new rectangle is $(x + 1)[(x + 2) + 3] = (x + 1)(x + 5)$, and the area of the old rectangle is $x(x + 2)$. (See Figures). Substituting these values into equation (1), we have

$$(x + 1)(x + 5) = 2[x(x + 2)].$$
$$x^2 + 6x + 5 = 2(x^2 + 2x)$$
$$x^2 + 6x + 5 = 2x^2 + 4x$$
$$6x + 5 = x^2 + 4x$$
$$5 = x^2 - 2x$$
$$x^2 - 2x - 5 = 0.$$

To find the roots of an equation in the form $ax^2 + bx + c = 0$ we use the quadratic formula

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In our case $a = 1$, $b = -2$, $c = -5$. Re-

placing these values in the quadratic formula we have

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2}$$

$$= \frac{2 \pm \sqrt{4 + 20}}{2}$$

$$= \frac{2 \pm \sqrt{24}}{2}$$

$$= \frac{2 \pm \sqrt{4 \cdot 6}}{2}$$

$$= \frac{2 \pm \sqrt{4}\ \sqrt{6}}{2}$$

$$= \frac{2 \pm 2\sqrt{6}}{2}$$

$$= \frac{2(1 \pm \sqrt{6})}{2}$$

$$= 1 \pm \sqrt{6}$$

Since $\sqrt{6} \cong 2.45$

$x = 1 + 2.45 + 3.45$ and $x = 1 - 2.45 = -1.45$.

We reject the negative value, as there are no rectangles with negative sides. Thus, the width of the original rectangle, x, is 3.45 ft., and the length, x + 2, is 5.45 ft.

Check: The area of rectangle I is $x(x + 2)$. Replacing x by 3.45 we have

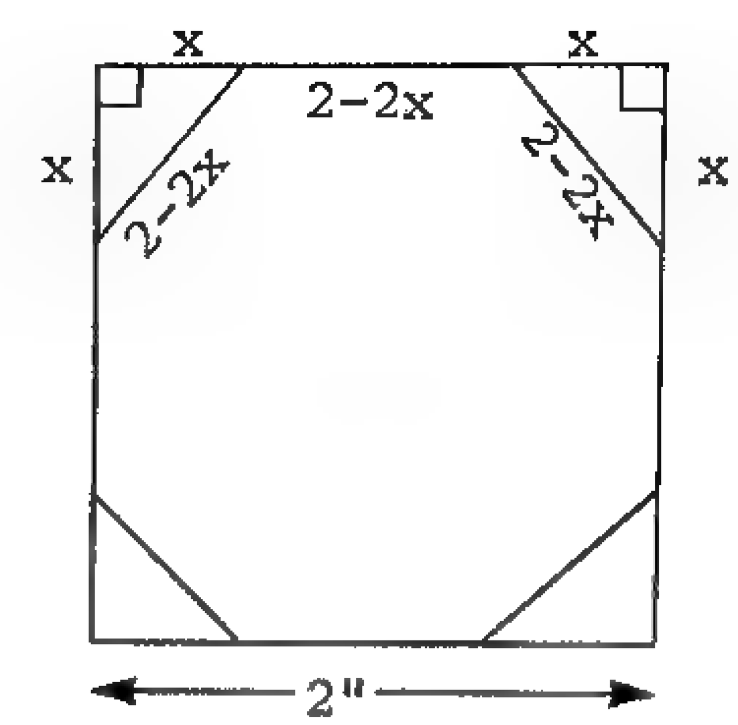
$$x(x + 2) = 3.45(3.45 + 2) = (3.45)(5.45) = 18.80.$$

The area of rectangle II is $(x + 1)[(x + 2) + 3]$. Replacing x by 3.45 we have

$$(x + 1)[(x + 2) + 3] = (3.45 + 1)(3.45 + 5) = (4.45)(8.45)$$
$$= 37.60$$
$$37.60 = 2(18.80).$$

Thus area rectangle II = 2(area rectangle I), and 3.45 is the correct value of x.

The corners of a 2-inch square are cut off so as to form a regular octagon. What is the length of the piece to be cut off?



Solution: In the figure we let x equal the length to be cut off. From the Figure it is seen that the side of the octagon is equal to the side of the square minus the lengths of two of the legs of the triangles that are cut off. Therefore, the side of the octagon is equal to $2 - x - x$ or $2 - 2x$. Since the small triangles formed at each corner are right triangles, we have, from the Pythagorean theorem,

$$x^2 + x^2 = (2 - 2x)^2 \qquad (1)$$

Equation 1 reduces to

$$x^2 - 4x + 2 = 0 \qquad (2)$$

Using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

with $a = 1$, $b = -4$, $c = 2$, the roots of Equation 2 are

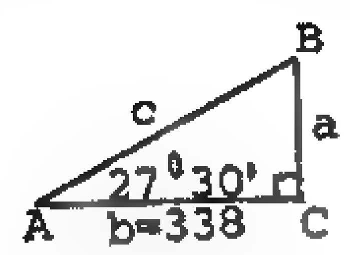
$x = 2 - \sqrt{2}$, and $x = 2 + \sqrt{2}$.

We must reject the second root, $2 + \sqrt{2}$, since the part cut off cannot be greater than the entire side. Therefore, the size of the piece cut off is $2 - \sqrt{2}$, or approximately 0.586 inches.

ANGLES OF ELEVATION, DEPRESSION AND AZIMUTH

• PROBLEM 12-21

From a point 338 feet from the base of a monument, and in a horizontal plane, the angle of elevation to the top is $27°30'$. Find the height of the monument.



Solution: In the figure, side a represents the height of the monument. Side b represents the horizontal distance (given as 338 feet) from the base of the monument. Angle BAC represents the angle of elevation, $27°30'$. Since we are given an adjacent side of the angle, side b, and are asked to find an opposite side, side a, we use the Tangent Function. Thus,

$$\tan A = \frac{a}{b};\ \tan 27°30' = \frac{a}{338}$$

Multiply both sides by 338. Then,

$$a = 338 \tan 27°30'$$

Looking at our trig table we find

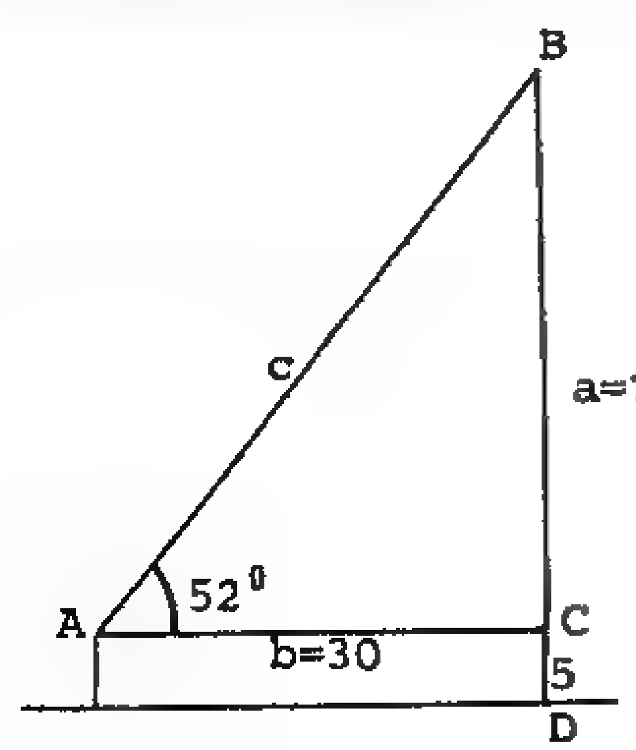
$$\text{Tan } 27°30' = 0.5206$$

therefore, $$a = 338(0.5206) = 176\ .$$

The monument is 176 feet tall.

• PROBLEM 12-22

From a point 5 ft. above the horizontal ground, and 30 ft. from the trunk of a tree, the line of sight to the top of the tree is measured as $52°$ with the horizontal. Find the height of the tree.



Solution: See the accompanying figure. A is taken at the observer's eye, B is the top of the tree, and the required height is BD = a + 5 ft. One relation connecting the known angle $A = 52°$, the known distance $b = 30$ ft., and the unknown length a is

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$

We therefore have, with the aid of trigonometric tables,

$$a = b \tan A = 30 \tan 52°\ .$$

We find in a table of trigonometric functions that $\tan 52°$ is 1.280. Then

$$a = 30 \times 1.280 = 38.4 \text{ ft.};$$

and the height of the tree is

$$BD = a + 5 = 43.4 \text{ ft.}$$

This result may be checked by first finding c from the relation cos A =

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

and then getting a from $\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$

$$\cos A = \frac{b}{c}$$

$$\cos 52° = \frac{30}{c}$$

$$c = \frac{30}{\cos 52°} = \frac{30}{.6157} \approx 48.7$$

$$\sin A = \frac{a}{c}$$

$$\sin 52° = \frac{a}{48.7}$$

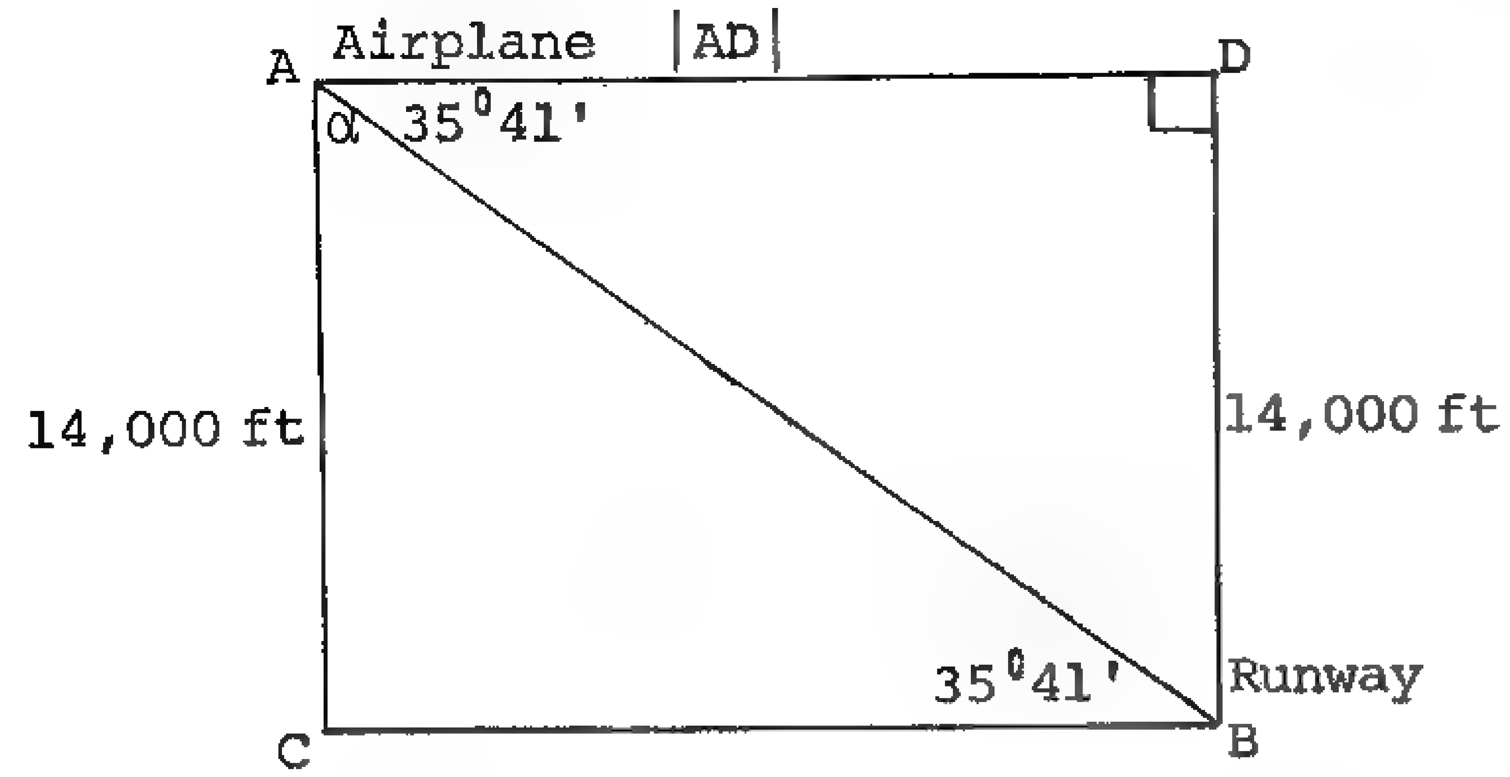
$$a = 48.7 (\sin 52°) = 48.7(.7880)$$

$$a \approx 38.4$$

• PROBLEM 12-23

From the cockpit of an airplane flying over level ground, the pilot sights the runway of a small airport, and the angle of depression of the runway is $35°41'$. If the plane is 14,000 ft above the surface, what is the distance to a point 14,000 ft directly above the runway? (Neglect the curvature of the surface of the earth.

Solution: From the figure we note that the distance AD is required. Triangle ADB is a right triangle; and since AD is parallel to CB, DB = 14,000; and we have a right triangle with one side and one acute angle given. The solution of the entire triangle is not needed- only one side. Thus, the side $|AD|$ can be found by computing the tan of the angle opposite the given side.



$$\tan = \frac{\text{side opposite the angle}}{\text{side adjacent the angle}} .$$

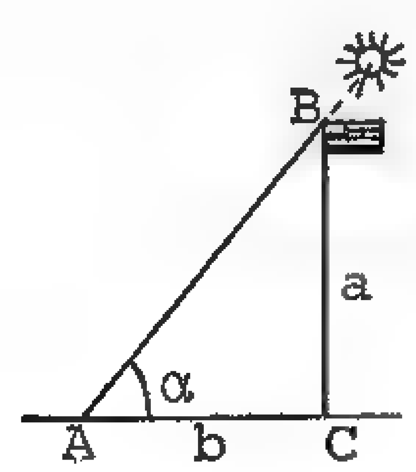
In this case the unknown is the adjacent side.

$$\tan 35^\circ 41' = \frac{|DB|}{|AD|}$$

$$|AD| = \frac{|DB|}{\tan 35^\circ 41'} = \frac{14{,}000'}{0.71813} = 19{,}495'$$

• PROBLEM 12-24

> If the angle of elevation of the sun is 49°27' find the height of a flagpole whose horizontal shadow is 83.59 feet.

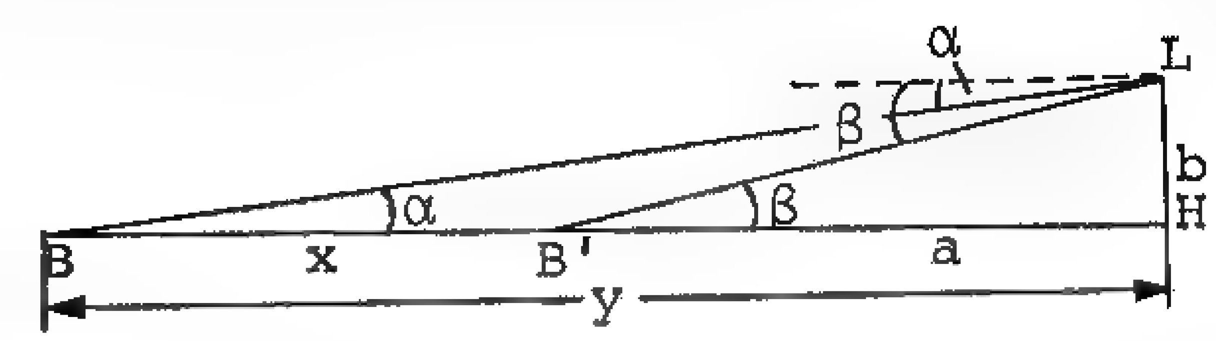


Solution: Construct the figure shown. BC represents the flagpole; AC, the shadow. The angle of elevation of the sun is at A; we represent it by α. We let AC = b and solve for BC = a.

In △ABC: b = 83.59; α = 49°27'.		
$\frac{a}{b} = \tan\alpha$, or $a = b\tan\alpha$	log b = 1.92215 log tan α = 0.06773 log a = 1.98988	a = 97.70 ft.

• PROBLEM 12-25

> From the top of a lighthouse 212 feet above a lake, the keeper spots a boat sailing directly towards him. He observes the angle of depression of the boat to be 6°13' and then later to be 13°7'. Find the distance the boat has sailed between the observations.

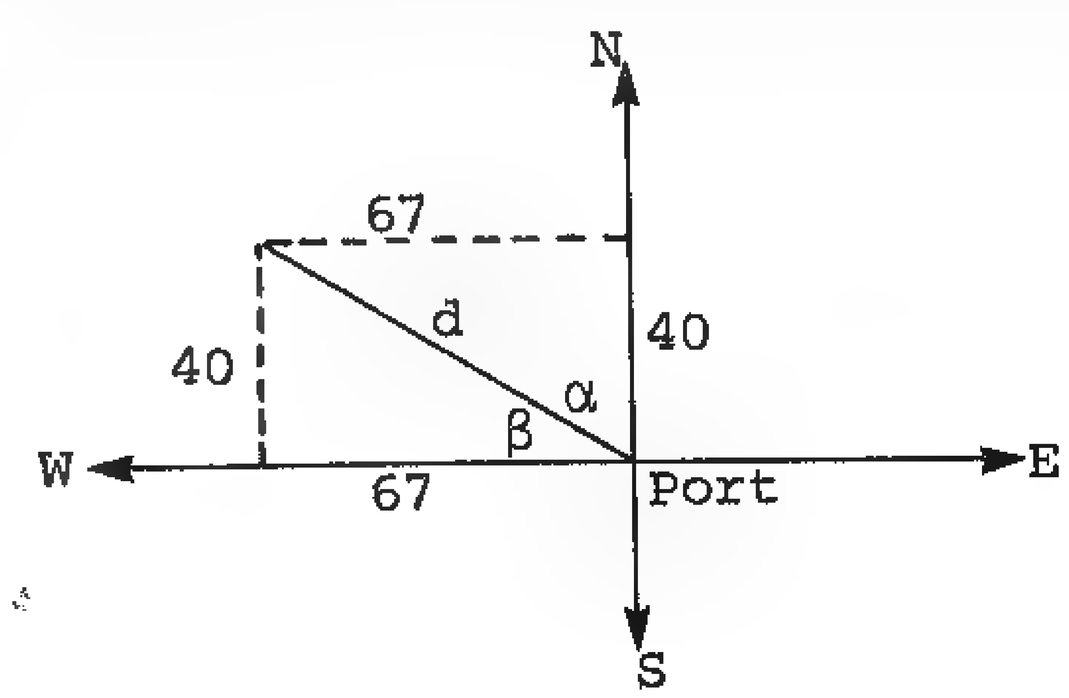


Solution: Construct the figure shown. B and B' represent the two positions of the boat. LH represents the lighthouse; angle α, the first angle of depression and β, the second. The line BH is parallel to the horizontal and therefore the angle at B equals α and the angle at B' equals β. We letter the figure as shown.

In △B'LH: b = 212; β = 13°7'.		
$\frac{a}{b} = \cot\beta$, or $a = b\cot\beta$.	log b = 2.32634 log cot β = 0.63262 log a = 2.95896	a = 909.8
In △BLH: b = 212; α = 6°13'.		
$\frac{y}{b} = \cot\alpha$, or $y = b\cot\alpha$.	log b = 2.32634 log cot α = 0.96286 log y = 3.28920	y = 1946.3
x = y − a = 1946.3 − 909.8 = 1036.5 ft.		

• PROBLEM 12-26

> A ship is 67 mi west and 40 mi north of a port. What is the distance and bearing of the ship from the port? (See Figure).



Solution: In order to specify the bearing of some traveling vehicle, the angle of the direction in which the vehicle is traveling must be discovered. To do this we superimpose our direction upon the coordin-

ate axes by extending a line from the origin to the point (x,y) where x is the distance traveled east or west, and y is the distance traveled north or south. The distance in the direction of travel is determined using the Pythagorean theorem: $d^2 = x^2 + y^2$. The angle that the direction makes with the east/west line is determined using the form

$$\tan \beta = \frac{y}{x} = \frac{\text{side opposite } \beta}{\text{side adjacent } \beta} .$$

The angle that the direction makes with a north/south line is determined using the form

$$\cot \beta = \frac{\text{side adjacent } \beta}{\text{side opposite } \beta} = \frac{x}{y}$$

or

$$\tan \alpha = \frac{\text{side opposite } \alpha}{\text{side adjacent } \alpha} = \frac{x}{y}$$

$$\tan \alpha = \frac{67}{40} = 1.6750 = \tan 59^\circ 9.7'$$

Hence

$$\alpha = 59^\circ 9.7'$$

and the bearing (azimuth) of the ship is N59°9.7'W.

$$d = \sqrt{(67)^2 + (40)^2} = \sqrt{4489 + 1600} = \sqrt{6089}$$

$$d = 78.032$$

MOTION

• PROBLEM 12-27

Mike can throw a football exactly 36 yards and Danny can throw a football exactly 30 yards. If they start to throw their football from the same spot, what is the minimum number of throws for each of them so that the footballs will have been thrown the same distance? What is this distance?

Throws:	1st.	2nd	3rd	4th	5th	6th
Mike	36	72	108	144	180	
Danny	30	60	90	120	150	180

Solution: Each time Mike throws the football, it travels a multiple of 36 yards and Danny's football travels a multiple of 30 yards. The figure illustrates what happens.

Multiples of 36 yards are the following:

$(1)(36) = 36$
$(2)(36) = 72$
$(3)(36) = 108$
$(4)(36) = 144$
$(5)(36) = 180$

Multiples of 30 yards are the following:

$(1)(30) = 30$

$(2)(30) = 60$
$(3)(30) = 90$
$(4)(30) = 120$
$(5)(30) = 150$
$(6)(30) = 180$

From the information in the figure, Mike must throw the football a minimum number of 5 times and Danny must throw it a minimum number of 6 times so that the footballs will be thrown the same distance, this same distance being 180 yards.

• PROBLEM 12-28

Two cars traveled the same distance. One car traveled at 50 mph and the other car traveled at 60 mph. It took the slower car 50 minutes longer to make the trip. How long did it take the faster car to make the trip?

Solution: Step. 1. Read problem again. Step 2. If we let t represent the number of hours the faster car travels, we can construct the following table from the given statements.

Note: The 50 minutes must be converted to $\frac{5}{6}$ hr.

This is done by the following:

$$\frac{50 \text{ minutes}}{60 \text{ minutes}} \times 1 \text{ hour} = \frac{50}{60} \text{ hours} = \frac{5}{6} \text{ hour.}$$

Step 3

	Distance	Rate	Time
Faster car	D	60 mph	t
Slower car	D	50 mph	t + 5/6

Note: Since the two cars travelled the same distance, the distance for both cars is D, as indicated in the above table.

Step 4

Formula $D = rt$,

where D is distance, r is rate, and t is time.

$$D=60t$$

$$D = 50\left(t + \frac{5}{6}\right)$$

Since the distances are the same, we can set the two expressions for D equal as in Step 5.

Step 5
$$60t = 50\left(t + \frac{5}{6}\right) \qquad (1)$$

Multiply each term within the parentheses by 50 to eliminate the parentheses.

$$60t = 50t + \frac{250}{6} \qquad (2)$$

Subtract 50t from both sides of equation (2),

$$60t - 50t = 50t + \frac{250}{6} - 50t$$

Therefore: $60t - 50t = \frac{250}{6}$.

Therefore: $10t = \frac{250}{6} \qquad (3),$

Multiply both sides of equation (3) by $\frac{1}{10}$.

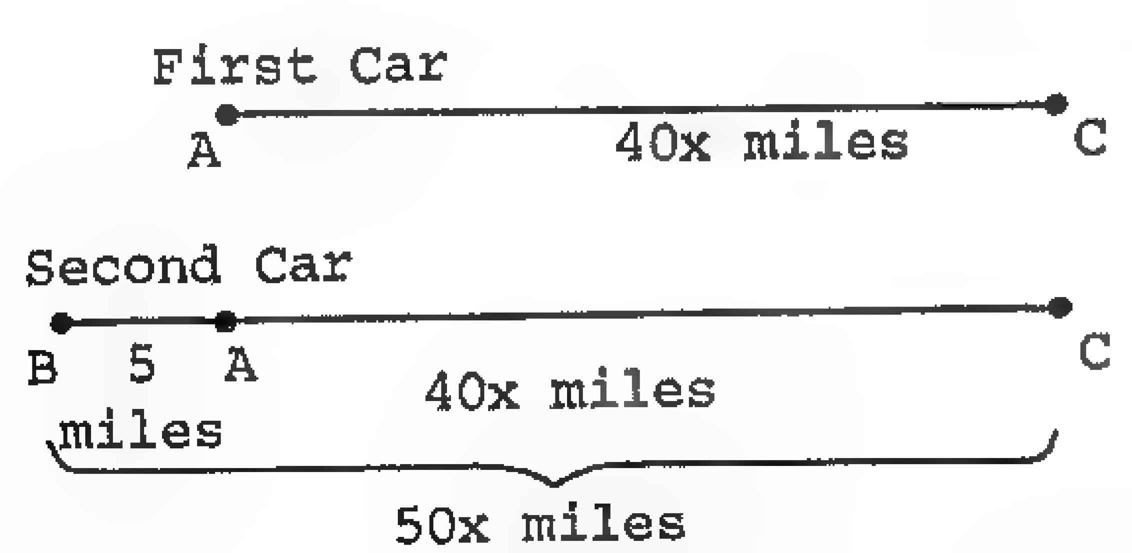
$$\frac{1}{\cancel{10}}(\cancel{10}t) = \frac{1}{\cancel{10}_{1}}\left(\frac{\cancel{250}^{25}}{6}\right)$$

$$t = \frac{25}{6} = 4\frac{1}{6} \text{ hours} = 4 \text{ hours } 10 \text{ minutes.}$$

Thus, it took the faster car 4 hours 10 minutes to make the trip.

• PROBLEM 12-29

Two cars are traveling 40 and 50 miles per hour, respectively. If the second car starts out 5 miles behind the first car, how long will it take the second car to overtake the first car?



	Rate (in mph)	Time (hours)	Distance (miles)
First Car	40	x	40x
Second Car	50	x	50x

<u>Solution:</u> Let x = number of hours it takes the second car to overtake the first car. See table.

Distance = rate × time

Then $\qquad 50x$ = distance second car travels in x hours,

and $\qquad 40x$ = distance first car travels in x hours.

Since the second car must travel an additional 5 miles (from B to A in diagram), $\quad 40x + 5 = 50x$

Simplify,
$$-10x = -5$$

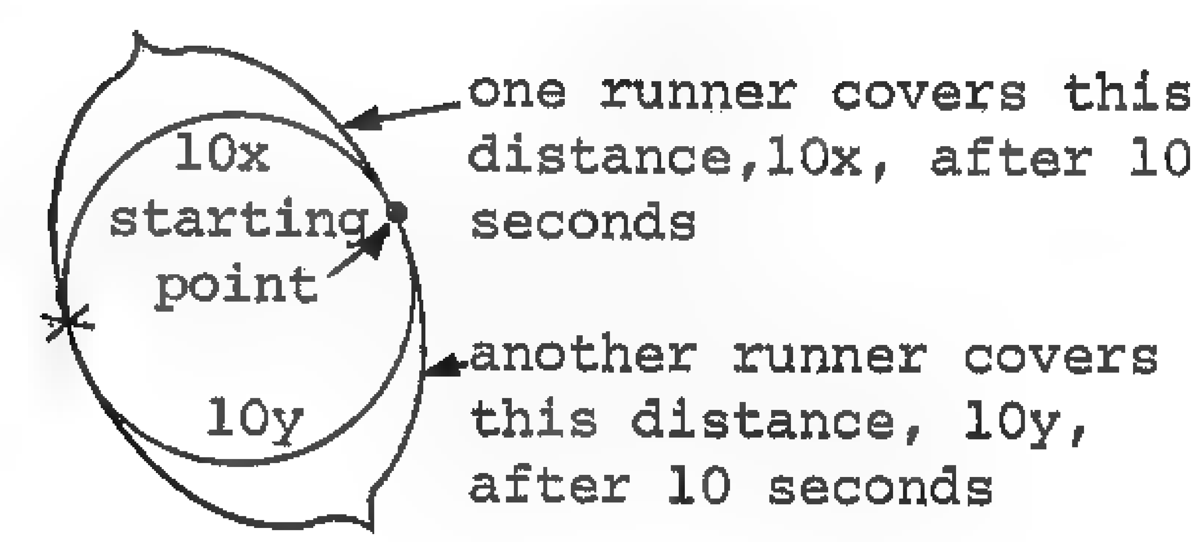
Divide by -10, $\quad x = \frac{1}{2}$, number of hours it takes the second car to overtake the first car.

Check: $\quad 40\left(\frac{1}{2}\right) + 5 = 50\left(\frac{1}{2}\right),$

$$25 = 25.$$

• PROBLEM 12-30

Two trackmen are running on a circular race track 300 feet in circumference. Running in opposite directions, they meet every 10 seconds. Running in the same direction, the faster passes the slower every 50 seconds. Find their rates in feet per second.



<u>Solution:</u> Let x = rate of the faster runner and y = rate of the slower runner. Using the formula

distance = rate × time

the first equation is:

$$10x + 10y = 300 \qquad (1)$$

This is true because both runners together cover the entire distance. (See figure.)

When they run in the same direction, the faster runner must gain 300 feet each time he overtakes the slower runner. Therefore, the distance that the faster runner covers in 50 sec, 50x, is equal to the circumference plus the distance of the slower runner, 50y. That is

$$50x = 300 + 50y$$

$$50x - 50y = 300 \qquad (2)$$

Equations 1 and 2 may be simplified, so that we have

$$x + y = 30$$

$$x - y = 6$$

Add these two equations and solve for x. The solution of this system is

$$x = 18$$

$$y = 12$$

The rates are therefore 18 feet per second for the faster runner and 12 feet per second for the slower runner.

To check the solution, we note that in 10 seconds the runners travel a total of 180 feet + 120 feet, or 300 feet, the length of the track. When they run in the same direction, in 50 seconds the faster travels 900 feet and the slower travels 600 feet, so that the faster runner runs 300 feet more than the slower, and therefore catches up with him every 50 seconds.

• PROBLEM 12-31

One plane flies at a ground speed 75 miles per hour faster than another. On a particular flight, the faster plane requires 3 hours and the slower one 3 hours and 36 minutes. What is the distance of the flight?

Solution: If the velocity of the fast plane is v_1 and that of the slow plane is v_2, we have

$$v_1 = 75 + v_2 \qquad (1)$$

since the fast plane is 75 miles per hour faster. The distance is the same for each plane. The distance is $d = v \cdot t$, where v is the velocity and t is the time. Note that the time for the slower plane expressed in hours is

$$3\frac{36}{60} = 3\frac{3}{5} \text{ hours.}$$

Hence,

$$3v_1 = 3\frac{3}{5}v_2 \qquad (2)$$

Equations (1) and (2) constitute a system of two equations in two variables. Solving equation (2) for v_1, we have

$$v_1 = \frac{6}{5}v_2$$

Substituting this value in equation (1), we have

$$\frac{6}{5}v_2 = 75 + v_2$$

$$\frac{1}{5}v_2 = 75$$

$$v_2 = 375$$

$$v_1 = 75 + 375 = 450$$

Hence, the length of the trip is $3 \cdot 450 = 1{,}350$ miles.

Check: The fast plane, velocity 450 miles per hour, is 75 miles per hour faster than the slow one, velocity 375 miles per hour. In 3 hours the fast plane travels $3 \cdot 450 = 1{,}350$ miles. In 3 hours 36 minutes the slow plane travels the same distance; that is, $3\frac{3}{5} \cdot 375 = 1{,}350$ miles.

• PROBLEM 12-32

An airplane with an air speed of 300 miles per hour flies against a head wind, and flies 900 miles in 4 hours. What is the speed of the head wind?

Solution: Let x = number of miles per hour in the rate of the head wind.

Then since the head wind is detracting from the speed of the airplane, let

$300 - x$ = number of miles per hour in the ground speed of the airplane.

Applying the formula, Rate $\times$ Time = Distance, with rate = $(300 - x)$,

Time = 4, Distance = 900, we obtain:

$$(300 - x)(4) = 900$$

Distributing 4, we have:

$$1200 - 4x = 900.$$

Subtract 1200 from both sides:

$$-4x = -300.$$

Divide by -4, $\quad x = 75$, number of miles per hour in the rate of the head wind.

Check: $\quad 4(300 - 75) = 900,$

$$900 = 900.$$

• PROBLEM 12-33

> Two planes with speeds of 600 miles (in still air) each make a trip of 990 miles. They take off at the same time and fly in opposite directions. One has a head wind and the other a tail wind. The plane flying with a tail wind lands 20 minutes before the other plane. What is the wind velocity?

Solution: If we let r represent the wind velocity in miles per hour, then the speeds of the planes are:

$600 + r$ speed of plane with tail wind (faster plane)

$600 - r$ speed of plane with head wind (slower plane).

Solving the distance formula,

distance = (rate)(time) for time, we obtain

time = $\dfrac{\text{distance}}{\text{rate}}$. We know distance is 990, rate for slower plane is $(600 - r)$, and rate for faster is $(600 + r)$. Thus, the time for the slower plane is

$$\frac{\text{distance}}{\text{rate}} = \frac{990}{600 - r}$$

and for the faster plane the time is

$$\frac{\text{distance}}{\text{rate}} = \frac{990}{600 + r}$$

The difference in time is 20 minutes. In hours it is $\frac{1}{3}$ hour. The time equation is:

$$\frac{990}{600 - r} - \frac{990}{600 + r} = \frac{1}{3}$$

Eliminate the fractions by multiplying both sides of the equation by $3(600 + r)(600 - r)$.

$$3(990)(600 + r) - 3(990)(600 - r)$$

$$= \frac{1}{3}(3)(600 + r)(600 - r)$$

$$3(990)[(600 + r) - (600 - r)] = (600 + r)(600 - r)$$

$$3(990)(2r) = 360{,}000 - r^2$$

$$r^2 + 5940r - 360{,}000 = 0$$

$$(r - 60)(r + 6000) = 0$$

$r - 60 = 0$	$r + 6000 = 0$
$r = 60$ mph	$r = -6000$ mph

The value -6000 mph cannot be considered a solution since it was implied in the problem that the wind velocity was positive. Also a wind speed of 6000 mph is extremely unlikely.

Check: If the wind speed is 60 mph., the planes will have velocities of $600 - 60 = 540$ mph and $600 + 60 = 660$ mph. The difference of the times of flight is

$$\frac{990}{600 - r} - \frac{990}{600 + r} = \frac{1}{3}, \text{ and substituting:}$$

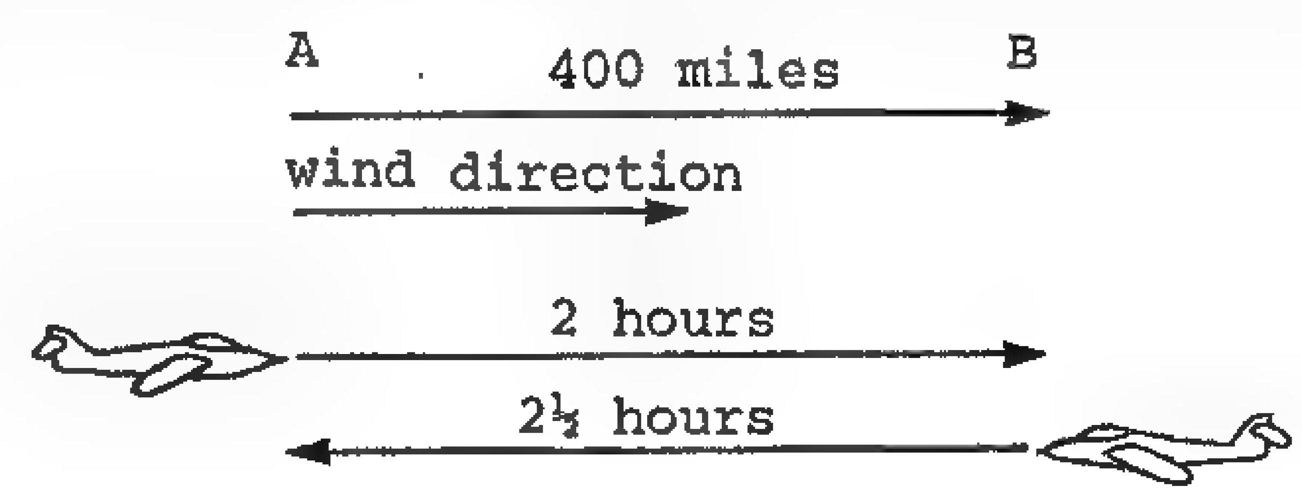
$$\frac{990}{540} - \frac{990}{660} = \frac{11}{6} - \frac{3}{2}$$

$$= \frac{11}{6} - \frac{9}{6}$$

$$= \frac{1}{3} \text{ hour.}$$

• PROBLEM 12-34

> Two airfields A and B are 400 miles apart, and B is due east of A. A plane flew from A to B in 2 hours and then returned to A in $2\frac{1}{2}$ hours. If the wind blew with a constant velocity from the west during the entire trip, find the speed of the plane in still air and the speed of the wind.



Solution: The essential point in solving this problem is that the wind helps the plane in flying from A to B and hinders it in flying from B to A. We therefore have the basis for two equations that involve the speed of the plane, the speed of the wind, and the time for each trip. We let

x = speed of plane in still air, in miles per hour
y = speed of wind, in miles per hour

Then, since the wind blew constantly from the west,

$x + y$ = speed of plane from A to B (wind helping)
$x - y$ = speed of plane from B to A (wind hindering)

The distance traveled each way was 400 miles, and so we have the following equations based on the formula distance/rate = time:

$$\frac{400}{x+y} = 2 = \text{time required for eastward trip} \qquad (8)$$

$$\frac{400}{x-y} = 2\tfrac{1}{2} = \text{time required for westward trip} \qquad (9)$$

We solve these equations simultaneously for x and y

$400 = 2x + 2y$	multiplying (8) by $x + y$		(10)
$800 = 5x - 5y$	multiplying (9) by $2(x - y)$		(11)
$2{,}000 = 10x + 10y$	multiplying (10) by 5		(12)
$\underline{1{,}600 = 10x - 10y}$	multiplying (11) by 2		(13)
$3{,}600 = 20x$	adding equations (12) and (13)		
$x = 180$	solving for x		
$400 = 360 + 2y$	replacing x by 180 in (10)		
$2y = 40$			
$y = 20$			

Therefore the solution set of equations (8) and (9) is $\{(180,20)\}$, and it follows that the speed of the plane in still air is 180 miles per hour and the speed of the wind is 20 miles per hour.

Check: $\dfrac{400}{180 + 20} = \dfrac{400}{200} = 2$ from (8)

$\dfrac{400}{180 - 20} = \dfrac{400}{160} = \dfrac{5}{2} = 2\tfrac{1}{2}$ from (9)

• PROBLEM 12-35

> A plane is heading in a direction 45° east of north and flying at an air speed (speed in still air) of 350 miles per hour. The wind is blowing in a direction 10° east of north at the rate of 70 miles per hour. Find the velocity (speed and direction) of the plane.

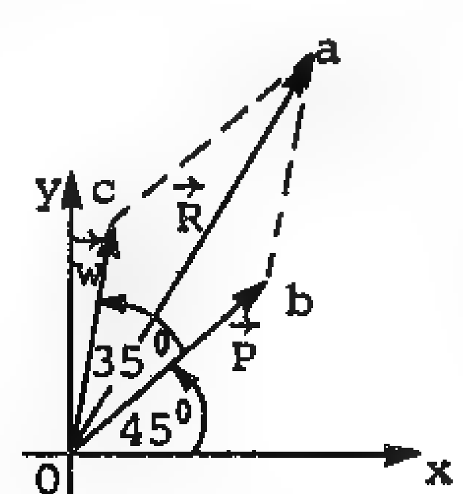


Fig. 1

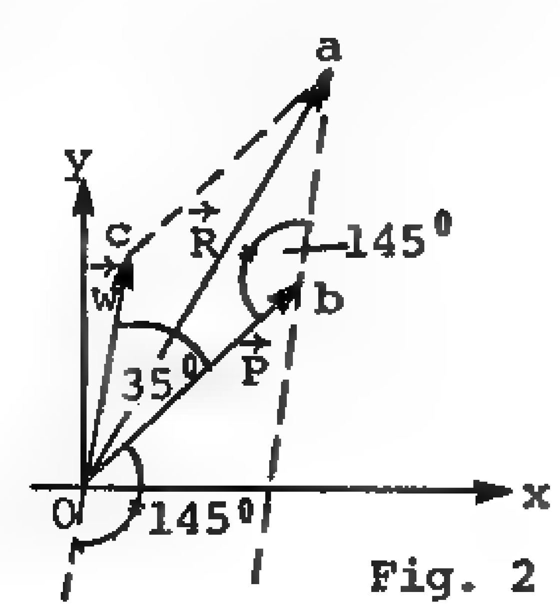


Fig. 2

Solution: Two velocities are affecting the flight of the plane, that due to the plane itself and that due to

the wind.

Velocity is a vector quantity and therefore the resultant or sum of two vectors is obtained using the parallelogram law for addition of vectors.

Let $\vec{P}$ represent the velocity of the plane

Let $\vec{W}$ represent the velocity of the wind.

Draw $\vec{P}$ at a 45° angle to the x axis and $\vec{W}$ 10° to the right of north or at an 80° angle to the x axis.

Now complete the parallelogram by drawing parallel sides to $\vec{P}$ and $\vec{W}$. Then connect the opposite vertices with a diagonal. Call this resultant $\vec{R}$. (See fig. 1)

R can now be determined using the law of cosines

$c^2 = a^2 + b^2 - 2ab \cos$ (included angle).

The included angle (oba) is determined as follows: $\vec{W}$ makes an angle of 80° and $\vec{P}$ makes an angle of 45° (with the x axis) so subtracting the angle of $\vec{P}$ from the angle of $\vec{W}$ gives the angle of (cob). Together, cob and oba must add up to 180° because they are supplemenarty angles. Angle cob equals 35° and 35° + oba = 180° or oba = 145° (see fig.2) . We now have

$R^2 = (P)^2 + (W)^2 - 2PW \cos 145°$ where the symbols without arrows above them represent the magnitudes of the corresponding vectors.

$$(R)^2 = (P)^2 + (W)^2 - [(2)(P)(W) \cos 145°]$$

$$(R)^2 = (350)^2 + (70)^2 - [(2)(350)(70)(-0.8192)]$$

Thus $R = 410$

By the law of sines, $\dfrac{\sin\angle aob}{70} = \dfrac{\sin 145°}{410}$

$$\sin\angle aob = \frac{70(0.5736)}{410} = 0.0979$$

$$\angle aob = 6° \quad \text{(nearest degree)}$$

Hence, the plane is flying 410 miles per hour in direction 39° east of north.

• PROBLEM 12-36

> The distance, s, a body falls in t seconds from a position of rest is given by the equation
>
> $$s = 16t^2$$
>
> where t is time in seconds, and s is measured in feet. A stone is dropped into a well and the sound of the splash is heard 3 seconds later. Taking the velocity of sound to be 1100 feet per second, write and solve the equation that determines the depth of the well.

Solution: From the formula, $s = 16t^2$, the time t_1 for the stone to reach the bottom of the well is

$$(t_1)^2 = \frac{s}{16}, \; t_1 = \frac{\sqrt{s}}{4}$$

From the formula, distance = rate X time, the time t_2 for the sound of the splash to reach the top of the well is

$$t_2 = \frac{\text{distance}}{\text{rate}} = \frac{s}{1100}$$

Since the total time is 3 seconds, we have the equation

$$t_1 + t_2 = \frac{\sqrt{s}}{4} + \frac{s}{1100} = 3$$

If we multiply each term of equation 4 by 1100, we obtain

$$275\sqrt{s} + s = 3300$$

This equation may be solved as follows. Let $u = \sqrt{s}$. Then $u^2 = s$ and substituting these values of s and $\sqrt{s}$ in the equation, we get

$$u^2 + 275u - 3300 = 0$$

By means of the quadratic formula

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ we find } u = 11.5$$

rejecting the negative root because it has no physical meaning. Since from the above equation, $s = u^2$,

$$s = 132.25$$

Therefore, the depth of the well is approximately 132.25 feet.

• PROBLEM 12-37

> How many gallons of a liquid that is 74 percent alcohol must be combined with 5 gallons of one that is 90 percent alcohol in order to obtain a mixture that is 84 percent alcohol?

Solution: If we let x represent the number of gallons needed of the first liquid and remember that 74 percent of x is 0.74x, then the table (see table) shows all the data given in this problem.

	Number of gallons	Percentage of alcohol	Number of gallons of alcohol
First liquid	x	74	0.74x
Second liquid	5	90	0.90(5) = 4.5
Mixture	x + 5	84	0.84(x + 5)

We are told that we are combining the number of gallons of alcohol in the 74 percent alcohol (0.74x) with the number of gallons of alcohol in the 90 percent alcohol (4.5) to obtain the number of gallons of alcohol in the 84 percent alcohol [0.84(x + 5)]. Thus

$$.74x + 4.5 = .84(x + 5)$$

Multiplying both sides by 100,

$$74x + 450 = 84\,(x + 5)$$
$$74x + 450 = 84x + 420$$
$$30 = 10x$$
$$x = 3$$

Therefore, 3 gallons of liquid that is 74 percent alcohol must be combined with 5 gallons of one that is 90 percent alcohol to obtain a mixture of 84 percent alcohol.

• PROBLEM 12-38

> How much water must be added to 500 gallons of alcohol that is 70 per cent pure to make a mixture that is 60 per cent pure?

Solution: Let x = number of gallons of water to be added.

Let $x + 500$ = number of gallons of liquid in the 60% pure mixture.

We can thus formulate the following chart:

AMOUNT OF LIQUID	× % ALCOHOL	= NUMBER OF GALLONS OF ALCOHOL
500 gallons	.70	500(.70)
500 + x gallons	.60	(500+x)(.60)

We form the equation:

$$(0.60)(500 + x) = (0.70)(500)$$

and we wish to solve for x.

Distribute, $300 + 0.60x = 350.$

Subtract 300 from both sides,

$$0.60x = 350 - 300.$$

Collect terms, $0.60x = 50.$

Divide by 0.60, $x = 83\frac{1}{3}$, number of gallons of water to be added.

• PROBLEM 12-39

> A grocer mixes two grades of coffee which sell for 70 cents and 80 cents per pound, respectively. How much of each must he take to make a mixture of 50 pounds which he can sell for 76 cents per pound?

Solution: Let x = the number of pounds of 70-cent coffee. Since the mixture is to contain 50 pounds and there are x pounds of 70 cent coffee, then $50 - x$ = number of pounds of 80 cent coffee. [Thus the total number of pounds in the mixture is x pounds (of 70-cent coffee) + $(50 - x)$ pounds (of 80-cent coffee) $= x - x + 50 = 50$ lbs, our desired amount]. Using our formula:

Number of Pounds		×Price per Pound (in cents)	= Total Price
70¢/lb. coffee	x	70	70x
80¢/lb. coffee	50-x	80	80(50-x)
76¢/lb.	50	76	76(50)

The total price of the 70¢ coffee, $70x$ and total price of the 80¢

coffee, $80(50 - x)$ equals total price of the 76¢ coffee, $(76)(50)$.

Therefore $70x + 80(50 - x) = (50)(76).$

Using the distributive law, we obtain:

$$70x + 4000 - 80x = 3800.$$

Subtract 4000 from both sides,

$$70x - 80x = 3800 - 4000.$$

Collect terms, $-10x = -200.$

Divide by -10, $x = 20$, number of pounds of 70-cent coffee.

Then $50 - x = 30$, number of pounds of 80-cent coffee.

Check: $(70)(20 + (80)(30) = (50)(76),$

$$1400 + 2400 = 3800,$$

$$3800 = 3800.$$

• PROBLEM 12-40

> How much water must be added to 5 gallons of 90% ammonia solution to reduce the solution to a 60% solution?

Solution: The volume of the original solution is 5 gallons. If we let x represent the number of gallons of water added to the original solution, then the final solution will have a volume of (5 + x) gallons. No ammonia is added so the only ammonia in the final solution is the ammonia in the original solution. We can now record our given information in tabular form:

	Total Volume (Gallons)	Percent Concentration	Gallons of 100% Ammonia
Original solution	5	90	4.5
Final solution	5 + x	60	4.5

Note: The gallons of 100% ammonia for the original solution was obtained in the following way: multiply the total volume of the original solution by the percent concentration of the original solution. Therefore,

$$5 \times 90\% = \cancel{5} \times \frac{90}{\cancel{100}_{20}} = \frac{90}{20} = \frac{9}{2} = 4.5 \text{ gallons of } 100\%$$

ammonia. Also, the gallons of 100% ammonia for the final solution is the same as the gallons of 100% ammonia for the original solution; that is, 4.5 gallons, since no ammonia was added.

The formula for the amount of substance in solution for the final solution is (total volume) X (percent concentration) = gallons of 100% ammonia. Therefore: $(5 + x) \times 60\% = 4.5$

or

$$(5 + x)\frac{60}{100} = 4.5$$

or $$(5 + x)\ .60 = 4.5$$

or $$.60(5 + x) = 4.5$$

$$3 + .60x = 4.5$$

$$.6x = 4.5 - 3$$

$$.6x = 1.5$$

$$x = \frac{1.5}{.6} = 2.5 \text{ gallons}$$

Therefore, 2.5 gallons of water must be added to reduce the concentration to 60%.

Check: If 2.5 gallons of water are added to the original solution, we will have 7.5 gallons of solution. The amount of 100% ammonia remains the same 4.5 gallons. The concentration equals

$$\frac{4.5}{7.5} \times 100 = \frac{450}{7.5} = 60\%.$$

• PROBLEM 12-41

How many gallons of a mixture containing 80% alcohol should be added to 6 gallons of a 25% solution to give a 30% solution?

Solution: Here we have

(1) How many gallons of the 80% mixture are needed?

(2) (alcohol in 80% solution) + (alcohol in 25% solution) = (alcohol in 30% solution).

These sentences become, in turn,

(1') Let x = number of gallons of the 80% solution;

(2') $0.80x + (0.25)(6) = 0.30(x + 6)$;

since the amount of alcohol in any solution is obtained by taking the total amount of the solution times the percentage of alcohol contained in it.

Solving for x from equation (2'):

Distributing, $0.80x + (0.25)(6) = 0.30x + (0.30)(6)$

$$0.80x - 0.30x = (0.30)(6) - (0.25)(6)$$

$$0.50x = (0.30 - 0.25)6$$

$$0.50x = (0.05)6$$

$$x = \frac{(0.05)6}{0.50} = \frac{(5)6}{50} = \frac{6}{10} = 0.6$$

Thus, 0.6 gallons of the 80% mixture are needed.

• PROBLEM 12-42

How many ounces of silver alloy which is 28% silver must be mixed with 24 ounces of silver which is 8% silver to produce a new alloy which is 20% silver?

Solution: Let x = number of ounces of 28% silver to be used. The relationship used to set up the equation is

Volume of 28% silver + Volume of 8% silver = Volume of silver in mixture.

$$.28x + .08(24) = .20(x + 24)$$
$$28x + 8(24) = 20(x + 24)$$
$$8x = 288$$
$$x = 36 \text{ ounces of silver}$$

Check: Volume of 28% silver = (.28)(36) = 10.08
Volume of 8% silver = (.08)(24) = 1.92
Total amount of silver = 12 ounces
The total mixture contains 24 + 36 = 60 ounces, and 12 ounces is 20% of 60 ounces.

• PROBLEM 12-43

How much water must be evaporated from 120 pounds of solution which is 3% salt to make a solution of 5% salt?

Solution: Let x = number of pounds of salt to be evaporated. The relationship used to set up the equation is

Amount of salt in new mixture = Amount of salt in old mixture.

$$.05(120 - x) = .03(120)$$
$$5(120 - x) = 3(120)$$
$$600 - 5x = 360$$
$$x = 48 \text{ pounds of water}$$

Check: Water in new mixture is 120 - 48 = 72 pounds. Of this 3% of 120 or 3.6 pounds is salt.

$$\frac{3.6}{72} = 5\%.$$

• PROBLEM 12-44

> Of 24 quarts of a mixture, 8% is iodine. Of another mixture, 4% is iodine. How many quarts of the second mixture should be added to the first mixture to obtain a mixture that is 5% iodine?

Solution: Let x = the number of quarts to be taken from second mixture.
The relationship used to set up the equation is:

Volume of iodine in the first mixture + Volume of iodine in the second mixture = Volume of iodine in the resulting mixture.

$$\begin{aligned} .08(24) + .04x &= .05(24 + x) \\ 1.92 + .04x &= 1.20 + .05x \\ .01x &= .72 \\ x &= 72 \text{ quarts of iodine} \end{aligned}$$

Check: Volume of iodine in the first mixture is (.08)(24) or 1.92 quarts. Volume of iodine in the second mixture is (.04)(72) or 2.88 quarts. Total volume of iodine is 4.80 quarts is equal to 5% of 96 quarts.

• PROBLEM 12-45

> A chemist has 24 ounces of a 25% solution of argyrol. How much water must he add to reduce the strength of the argyrol to 20%?

Solution: Let x = number of ounces of water to be added.
The relationship used to set up the equation is (since only water is added)

Amount of argyrol in new mixture = Amount of argyrol in old mixture.

$$\begin{aligned} .20(24 + x) &= .25(24) \\ 4.8 + .2x &= 6 \\ 2x &= 12 \\ x &= 6 \text{ ounces of water} \end{aligned}$$

Check: The new mixture has 30 ounces. Of this, 6 ounces or 20% of the mixture is argyrol.

• PROBLEM 12-46

> How many quarts of pure alcohol must be added to 40 quarts of a mixture that is 35% alcohol to make a mixture that will be 48% alcohol?

Solution: Let x = number of quarts of pure alcohol to be added.
The relationship used to set up the equation is

Amount of alcohol in new mixture = Amount of alcohol in old mixture + Amount of alcohol added

$$\begin{aligned} .48(40 + x) &= (.35)(40) + x \\ 19.2 + .48x &= 14 + x \\ x &= 10 \text{ quarts of alcohol} \end{aligned}$$

Check: Amount of alcohol in new mixture = 14 quarts + 10 quarts = 24 quarts. New mixture contains a total of 40 + 10 = 50 quarts.

$$\frac{24}{50} = 48\%.$$

• PROBLEM 12-47

> A truck radiator contains 32 quarts of a 20% solution of anti-freeze. How much of the original solution must be drawn off and replaced by pure anti-freeze to obtain a solution of 45% anti-freeze?

Solution: Let x = number of quarts of solution drawn off.
The relationship used to set up the equation is

Volume of anti-freeze left in 20% solution after drawing off x quarts + Volume of anti-freeze added = Volume of anti-freeze in 45% solution.

$$\begin{aligned} .20(32 - x) + x &= .45(32) \\ 6.4 - .20x + x &= 14.40 \\ .80x &= 8 \\ x &= 10 \text{ quarts of anti-freeze added} \end{aligned}$$

Check: The original mixture contained 32 quarts. From this 10 quarts were drawn off. Of this 20% or 2 quarts were anti-freeze. Since the original mixture contained 20% of 32 quarts or 6.4 quarts of anti-freeze, this left 4.4 quarts of anti-freeze. To this was added 10 quarts of anti-freeze making a total of 14.4 quarts of anti-freeze. 14.4 quarts is 45% of 32.

• PROBLEM 12-48

> A storekeeper has two kinds of cookies, one worth \$.75 a pound and the other worth \$.50 a pound. How many pounds of each should he use to make a mixture of 60 pounds worth \$.55 a pound?

Solution: Let x = number of pounds of cookies at 75 cents. Then (60 - x) = the number of cookies at 50 cents a pound.

The relationship used to set up the equation is:

Value of 75 cent cookies + Value of 50 cent cookies = Value of mixture.

$$75x + 50(60 - x) = 60(55)$$

$$75x + 3000 - 50x = 3300$$
$$25x = 300$$
$$x = 12 \text{ pounds of 75 cent cookies}$$
$$60 - x = 48 \text{ pounds of 50 cent cookies}$$

Check: The value of 12 pounds of cookies at 75 cents a pound is \$9.00. The value of 48 pounds of cookies at 50 cents a pound is \$24.00. The resulting mixture contains 60 pounds worth \$33.00. If 60 pounds of cookies are sold for \$33.00 then each pound is sold for \$.55.

• PROBLEM 12-49

If a container contains a mixture of 5 gallons of white paint and 11 gallons of brown paint, how much white paint must be added to the container so that the new mixture will be two-thirds white paint?

Solution: Let x = number of gallons of white paint to be added then x + 5 = number of gallons of white paint in the final mixture. x + 16 = total number of gallons of paint in the final mixture.

The new mixture will be two-thirds white paint if the ratio of the final number of gallons of white paint to the final total number of gallons of paint is equal to $\frac{2}{3}$. The proportion is

$$\frac{x + 5}{x + 16} = \frac{2}{3}$$

$3x + 15 = 2x + 32$ by cross-multiplying

$3x - 2x = 32 - 15$ by isolating x terms

$x = 17$ gallons by combining like terms.

Check: The final mixture would contain x + 5 = 22 gallons of white paint, and the total number of gallons of paint would be x + 16 = 33 gallons.

$$\frac{\text{final number of gallons of white paint}}{\text{final total number of gallons of paint}} = \frac{22 \text{ gal}}{33 \text{ gal}} = \frac{2}{3}.$$

• PROBLEM 12-50

A chemist has an 18% solution and a 45% solution of a disinfectant. How many ounces of each should be used to make 12 ounces of a 36% solution?

Solution: Let x = Number of ounces from the 18% solution
And y = Number of ounces from the 45% solution

(1) $x + y = 12$

(2) $.18x + .45y = .36(12) = 4.32$

Note that .18 of the first solution is pure disinfectant and that .45 of the second solution is pure disinfectant. When the proper quantities are drawn from each mixture the result is 12 gallons of mixture which is .36 pure disinfectant, i.e., the resulting mixture contains 4.32 ounces of pure disinfectant.

When the equations are solved, it is found that

$$x = 4 \quad \text{and} \quad y = 8.$$

• PROBLEM 12-51

A man wants to obtain 15 gallons of a 24% alcohol solution by combining a quantity of 20% alcohol solution, a quantity of 30% alcohol solution and 1 gallon of pure water. How many gallons of each of the alcohol solutions must be used?

Solution: Let x = The number of gallons of the 20% solution used
And y = The number of gallons of the 30% solution used

(1) $x + y + 1 = 15$

(2) $.20x + .30y = .24(15) = 3.6$

When these equations are solved it is found that 8 gallons of the 20% solution and 6 gallons of the 35% solution are used.

• PROBLEM 12-52

In a chemical laboratory one carboy contains 12 gallons of acid and 18 gallons of water. Another carboy contains 9 gallons of acid and 3 gallons of water. How many gallons must be drawn from each carboy and combined to form a solution that is 7 gallons acid and 7 gallons water?

Solution: Let x = Number of gallons taken from first carboy
And y = Number of gallons taken from second carboy

(1) $x + y = 14$

(2) $\frac{12x}{30} + \frac{9y}{12} = 7$

In forming the second equation, it should be observed that $\frac{12}{30}$ of the liquid drawn from the first carboy is acid and $\frac{9}{12}$ of the liquid drawn from the second carboy is acid. The two quantities of liquid drawn from the two carboys yield 7 gallons of acid in the mixture.

When the equations are solved, it is found that

$$x = 10 \quad \text{and} \quad y = 4$$

• **PROBLEM 12-53**

Sand and gravel have been mixed in two separate piles. In the first pile the ratio of sand to gravel is 1:1, and in the second pile the ratio of sand to gravel is 1:4. A third pile, in which the ratio of sand to gravel is 1:3, is to be formed from the first two piles. If the third pile is to contain 15 cubic yards, how many cubic yards must be taken from each of the first two piles?

Solution: Let x = Number of cubic yards to be taken from first pile
And y = Number of cubic yards to be taken from second pile

(1) $x + y = 15$

(2) $\frac{1}{2}x + \frac{1}{5}y = \frac{1}{4}(15)$

In forming the second equation it should be observed that when the ratio of sand to gravel in the first pile is 1:1 then $\frac{1}{2}$ of the material taken from the first pile is sand. Likewise, the ratio of sand to gravel in the second pile is 1:4 and therefore 1/5 of the material taken from the second pile is sand. Since the ratio of sand to gravel in the third pile is 1:3 then $\frac{1}{4}$ of the 15 cubic yards in the third pile is sand.

When the equations are solved it is found that $x = 2\frac{1}{2}$ and $y = 12\frac{1}{2}$.

• **PROBLEM 12-54**

What quantities of silver 60 per cent and 82 per cent pure must be mixed together to give 12 ounces of silver 70 per cent pure?

Solution: Let x = number of ounces of 60 per cent silver, and

y = number of ounces of 82 per cent silver.

We use the following table to describe the given information:

	Number of ounces	% Pure Silver	Number of Ounces of Pure Silver
Silver (60%)	x	60	.60x
Silver (82%)	y	82	.82y
Silver (70%)	12	70	.70(12)

From the information obtained in the table we have the following equations:

$$.60x + .82y = .70(12) \quad (1)$$

$$x + y = 12 \quad (2)$$

Multiplying each term of equation (1) by 100, we obtain:

$$60x + 82y = 70(12) \quad (3)$$

Equation (2) multiplied by 60 gives:

$$60x + 60y = (12)(60). \quad (4)$$

Then equation (3)-(4) gives:

$$\begin{array}{r} 60x + 82y = 840 \\ -60x - 60y = -720 \\ \hline 22y = 120 \end{array};$$

dividing both sides by 22, $\quad y = \frac{120}{22} = \frac{60}{11} = 5\frac{5}{11}$.

Substituting $5\frac{5}{11}$ for y in (2) gives

$$x + 5\frac{5}{11} = 12, \text{ or } x + \frac{60}{11} = \frac{132}{11}$$

Therefore, $\quad x = \frac{72}{11} = 6\frac{6}{11}$.

Thus, we must mix $6\frac{6}{11}$ ounces of 60 per cent pure silver and $5\frac{5}{11}$ ounces of 82 per cent pure silver to obtain 12 ounces of silver 70 per cent pure.

Check: Substituting $6\frac{6}{11}$ for x and $5\frac{5}{11}$ for y in (3) gives

$$\left(6\frac{6}{11}\right)(60) + \left(5\frac{5}{11}\right)(82) = (70)(12)$$

Convert $6\frac{6}{11}$ and $5\frac{5}{11}$ to fractions, $\frac{72}{11}(60) + \frac{60}{11}(82) = 840$.

Multiply, $\frac{4320}{11} + \frac{4920}{11} = 840$

Add fractions, $\frac{9240}{11} = 840$

$$840 = 840 .$$

Substituting in (2) gives

$$6\frac{6}{11} + 5\frac{5}{11} = 12$$
$$12 = 12.$$

• **PROBLEM 12-55**

A tobacco dealer mixed 12 pounds of one grade of tobacco with 10 pounds of another grade to obtain a blend worth \$54. He then made a second blend worth \$61 by mixing 8 pounds of the first grade with 15 pounds of the second grade. Find the price per pound of each grade.

Solution: In this problem we have two basic relations that we can use to form two equations. We let

x = price per pound of the first grade, in dollars

y = price per pound of the second grade, in dollars

The relationship used to set up the equations is:

Value of the first grade + value of the second grade = Value of the mixture; that is,

(Number of pounds of the first grade)(Price per pound of the first grade) + (Number of pounds of the second grade)(Price per pound of the second grade) = The total cost of the mixture. Then

$12x + 10y = 54$ (1) using the numbers of pounds as coefficients and the values of the blends as constant terms

$8x + 15y = 61$ (2)

We eliminate y by subtraction:

$36x + 30y = 162$	multiplying (1) by 3
$\underline{16x + 30y = 122}$	multiplying (2) by 2
$20x = 40$	equating the differences of the members
$x = 2$	solving for x
$16 + 15y = 61$	replacing x by 2 in (2)
$15y = 45$	
$y = 3$	

Therefore the solution set of equations (1) and (2) is $\{(2, 3)\}$, and it follows that the prices of the two grades are \$2 and \$3 per pound.

Check: To check the solution of a verbal problem, we reread the problem, substituting the values found and verify if they make the statements true.

$$12(2) + 10(3) = 24 + 30 = 54$$

$$8(2) + 15(3) = 16 + 45 = 61$$

NUMBERS, DIGITS, COINS AND CONSECUTIVE INTEGERS

• PROBLEM 12-56

The sum of two numbers is 23. One of the numbers is 7 more than the other number. What are the numbers?

Solution: Let x = one of the numbers, and $x + 7$ = the other number. Since we are given that the sum of the two numbers is 23,

$$x + (x + 7) = 23.$$

By the associative law of addition:

$$x + (x + 7) = 23$$

is the same as $(x + x) + 7 = 23,$

or $2x + 7 = 23.$

Subtract 7 from both sides:

$$2x = 23-7.$$

Collect terms, $2x = 16.$

Divide by 2, $x = 8$, one of the numbers.

Then solving for our other number $x + 7$, we substitute 8 for x.

Hence, $x + 7 = 8 + 7 = 15$, the other number.

Therefore, the two numbers are 8 and 15. We can verify this result by observing that the sum of the two numbers is indeed 23, and 15 is 7 more than 8, $8 + 7 = 15$.

• PROBLEM 12-57

The sum of a number and its reciprocal is $\frac{65}{28}$. What is the number?

Solution: Given a number x. Its reciprocal is written as $\frac{1}{x}$. The sum of x and its reciprocal $\frac{1}{x}$, that is $x + \frac{1}{x}$, equals $\frac{65}{28}$. We have: $x + \frac{1}{x} = \frac{65}{28}$.

Multiplying both sides of this equation by $28x$, the least common denominator, we obtain:

$$28x^2 + 28 = 65x$$
$$28x^2 - 65x + 28 = 0$$
$$(7x - 4)(4x - 7) = 0$$

$$7x - 4 = 0 \qquad\qquad 4x - 7 = 0$$

$$x = \frac{4}{7} \qquad\qquad x = \frac{7}{4}$$

Note: The two possible values of the number are reciprocals so a single check will suffice to show the number could be either $\frac{4}{7}$ or $\frac{7}{4}$.

Check: $\frac{4}{7} + \frac{7}{4} = \frac{16}{28} + \frac{49}{28} = \frac{65}{28}$.

• PROBLEM 12-58

The sum of two numbers is 24; one number is 3 more than twice the other. Find the numbers.

Solution: Let x = one of the numbers
Let y = the other number
Since the sum of the two numbers is 24,

$$x + y = 24 \qquad (1)$$

and since one of the numbers is 3 more than twice the other,

$$x = 2y + 3 \qquad (2)$$

Thus we have 2 equations in 2 unknowns and we solve for x and y:
Since $x = 2y + 3$, we may replace x by $(2y + 3)$ in equation (1),

$$(2y + 3) + y = 24$$
$$3y + 3 = 24$$
$$3y = 21$$
$$y = 7$$

To solve for x we replace y by 7 in equation (2)

$$x = 2(7) + 3$$
$$x = 14 + 3$$
$$x = 17$$

Thus the two numbers are 17 and 7.

Check: The sum of the two numbers is 24:

$$x + y = 17 + 7 = 24$$

One of the numbers is 3 more than twice the other:

$$17 = 2(7) + 3$$
$$17 = 14 + 3$$
$$17 = 17$$

• PROBLEM 12-59

The sum of two numbers is 25 and the difference of their squares is 225. Find the numbers.

Solution: Let x = one of the two numbers.
Let y = the other number.

Since the sum of the two numbers is 25,

$$x + y = 25 \qquad (1)$$

and since the difference of their squares is 225,

$$x^2 - y^2 = 225 \qquad (2)$$

Thus we now have 2 equations in 2 unknowns:

$$x + y = 25 \qquad (1)$$
$$x^2 - y^2 = 225 \qquad (2)$$

Solving equation (1) for y we obtain

$$y = 25 - x \qquad (3)$$

To solve for x, we substitute this value of y into equation (2),

$$x^2 - (25 - x)^2 = 225$$
$$x^2 - (625 - 50x + x^2) = 225$$
$$x^2 - 625 + 50x - x^2 = 225$$
$$50x - 625 = 225$$
$$50x = 850$$
$$x = 17$$

To solve for y, we substitute this value of x into equation (1)

$$17 + y = 25$$
$$y = 8$$

Thus the two numbers are 17 and 8.

Check: Replace x and y by 17 and 8 in equations (1) and (2):

$$x + y = 25 \qquad (1)$$
$$17 + 8 = 25$$
$$25 = 25$$
$$x^2 - y^2 = 225 \qquad (2)$$
$$(17)^2 - (8)^2 = 225$$
$$289 - 64 = 225$$
$$225 = 225$$

• PROBLEM 12-60

Find the number which, increased by its reciprocal, is equal to 37/6.

Solution: Let x = The number

Then $\frac{1}{x}$ = The reciprocal

$$x + \frac{1}{x} = \frac{37}{6}$$

$$6x^2 + 6 = 37x$$

$$6x^2 - 37x + 6 = 0$$

$$(6x - 1)(x - 6) = 0$$

$$6x - 1 = 0$$

$$x = \frac{1}{6}$$

$$x - 6 = 0$$

$$x = 6$$

The number may be taken as 1/6 and the reciprocal as 6, or the number may be taken as 6 and the reciprocal as 1/6.

Check: If the number is 6, the reciprocal is 1/6. Then

$$6 + \frac{1}{6} = \frac{37}{6}.$$

• PROBLEM 12-61

Find two numbers such that twice the first added to the second equals 19, and three times the first is 21 more than the second.

Solution: Let x = the first number and y = the second number. The equations are

$2x + y = 19$ (twice the first added to the second equals 19)

$3x = y + 21$ (three times the first is 21 more than the second)

To solve this system

$$2x + y = 19$$

$$3x = y + 21$$

obtain all the variables on one side of the equations.

$$2x + y = 19 \qquad (1)$$

$$3x - y = 21 \qquad (2)$$

Add (2) to (1)

$$2x + y = 19 \qquad (1)$$

$$\underline{3x - y = 21} \qquad (2)$$

$$5x \quad = 40 \qquad (3)$$

Divide by 5 to obtain x

$$x = 8$$

Substitute $x = 8$ into (1) or (2).

(1) $\quad 2x + y = 19$

$$2(8) + y = 19$$

$$16 + y = 19$$

$$y = 3$$

The solution of this system is

$x = 8$, the first number

$y = 3$, the second number

To check the solution, show that the two numbers satisfy the conditions of the problem.

Twice the first number is $2(8) = 16$. Add this result to the second is $16 + 3 = 19$. Thus $19 = 19$. Then three times the first number is $3(8) = 24$ which is 21 more than 3. That is $24 = 21 + 3$; $24 = 24$.

• PROBLEM 12-62

Find two numbers such that the sum of twice the larger and the smaller is 64. But, if 5 times the smaller be subtracted from four times the larger the result is 16.

Solution: Let x = the larger number
And y = the smaller number
Then $2x + y = 64$
$4x - 5y = 16$

When these equations are solved simultaneously the larger number is found to be 24 and the smaller number 16.

• PROBLEM 12-63

Separate 120 into two parts such that the larger exceeds three times the smaller by 12.

Solution: Let x = the larger number
And y = the smaller number
Then $x + y = 120$
$x = 3y + 12$

When these equations are solved simultaneously the larger number is found to be 93 and the smaller 27.

• PROBLEM 12-64

Find two real numbers whose sum is 10 such that the sum of the larger and the square of the smaller is 40.

Solution: Let x = the smaller number
Let y = the larger number

The sum of the numbers is 10, therefore

$$x + y = 10 \qquad (1)$$

The sum of the larger and the square of the smaller is 40, therefore

$$y + x^2 = 40 \qquad (2)$$

Solving for y in equation (1) by adding $(-x)$ to both sides we obtain

$$y = 10 - x \qquad (3)$$

Replacing this value of y in equation (2) we obtain

$$(10 - x) + x^2 = 40$$

Adding -40 to both sides,

$$10 - x + x^2 - 40 = 0$$

$$x^2 - x - 30 = 0$$

Factoring,

$$(x - 6)(x + 5) = 0$$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$. Thus, either

$$x - 6 = 0 \quad \text{or} \quad x + 5 = 0$$

and

$$x = 6 \quad \text{or} \quad x = -5.$$

To find the corresponding y values we replace x by each of these values in equation (3):

Replacing x by 6:

$$y = 10 - 6$$
$$y = 4$$

Replacing x by -5:

$$y = 10 -(-5)$$
$$y = 10 + 5$$
$$y = 15$$

Thus the two possible solutions are (6,4) and (-5,15). Since we assumed x to be the smaller number, and 6 is greater than 4, not smaller than it, we reject (6,4). To check if (-5,15) fits the conditions of this problem, we replace (x,y) by (-5,15) in equations (1) and (2):

$$x + y = 10 \qquad (1)$$
$$-5 + 15 = 10$$
$$10 = 10$$

$$y + x^2 = 40 \qquad (2)$$
$$15 + (-5)^2 = 40$$
$$15 + 25 = 40$$
$$40 = 40$$

Thus the pair of numbers whose sum is ten such that the sum of the larger and the square of the smaller is 40 is (-5,15).

• PROBLEM 12-65

If 3 is subtracted from the numerator of a certain fraction, the value of the fraction becomes 3/5. If 1 is subtracted from the denominator of the same fraction then the value of the fraction becomes 2/3. Find the original fraction.

Solution: Let x = the numerator of the fraction
And y = the denominator of the fraction

Then (1) $\dfrac{x - 3}{y} = \dfrac{3}{5}$, $5x - 15 = 3y$

(2) $\dfrac{x}{y - 1} = \dfrac{2}{3}$, $3x = 2y - 2$

(3) $5x - 3y = 15$

(4) $3x - 2y = -2$

When these equations are solved simultaneously x is found to be 36 and y is found to be 55. Therefore, the original fraction is 36/55.

• PROBLEM 12-66

The units digit of a two digit number is two larger than the tens digit. When the digits are reversed, the new two digit number is equal to seven times the sum of the digits. What is the original number?

Solution: When dealing with number problems that are concerned with the digits of a number, we must utilize position values and write a two digit number as $10a + b$, a three digit number as $100x + 10y + z$, and so on.

Let x represent the digit in the tens position, then the digit in the units position must be $x + 2$. The original two digit number is of the form $10a + b$ where a = digit in tens position and b = digit in the units position. Therefore, the original number is

$$10x + (x + 2) \quad \text{or} \quad 10x + x + 2 \quad \text{or} \quad 11x + 2.$$

When the digits are reversed, the new number can be expressed as

$$10(x + 2) + x \quad \text{or} \quad 11x + 20.$$

The sum of the digits is $x + (x + 2) = 2x + 2$. Setting the new number equal to seven times the sum of the digits, we have:

$$11x + 20 = 7(2x + 2)$$

$$11x + 20 = 14x + 14$$

$$20 - 14 = 14x - 11x$$

$$6 = 3x$$

$$x = 2, \ x + 2 = 4$$

The original number $10x + (x + 2) = 10(2) + 4 = 24$.

If the digits are reversed, we get 42, and this is the product of seven and the sum of the digits (six).

• PROBLEM 12-67

> The sum of the digits of a two-digit number is 9. The number is equal to 9 times the units' digit. Find the number.

Solution: (1) $t + u = 9$

(2) $10t + u = 9u$

If these two equations are solved simultaneously, $t = 4$ and $u = 5$ and the number is 45.

• PROBLEM 12-68

> The units' digit of a two-digit number is 4 less than 3 times the tens' digit. If the digits are reversed, a new number is formed which is 12 less than twice the original number. Find the number.

Solution: (1) $u = 3t - 4$

(2) $10u + t = 2(10t + u) - 12$

If these equations are solved simultaneously $t = 4$ and $u = 8$ and the number is 48.

• PROBLEM 12-69

> A purse contains 19 coins worth \$3.40. If the purse contains only dimes and quarters, how many of each coin are in the purse?

Solution: Let x = the number of dimes in the purse

Then $19 - x$ = the number of quarters in the purse
$10x$ = the value of the dimes
$25(19 - x)$ = the value of quarters

The relationship used in setting up the equation is:
The value of the dimes + the value of the quarters = \$3.40

$$10x + 25(19 - x) = 340$$
$$10x + 475 - 25x = 340$$
$$x = 9$$

There are 9 dimes and 10 quarters in the purse.

Check: The dimes are worth \$.90 and the quarters are worth \$2.50, making a total of \$3.40.

• PROBLEM 12-70

> A toy savings bank contains \$17.30 consisting of nickels, dimes, and quarters. The number of dimes exceeds twice the number of nickels by 3 and the number of quarters is 4 less than 5 times the number of nickels. How many of each coin are in the bank?

Solution: Let x = the number of nickels
Then $2x + 3$ = the number of dimes
And $5x - 4$ = the number of quarters
The relationship used in setting up the equation is:

Value of nickels + Value of dimes + Value of quarters = 1730.

$$5x + 10(2x + 3) + 25(5x - 4) = 1730; \ x = 12$$

There are 12 nickels, 27 dimes and 56 quarters in the bank.

Check: The nickels are worth \$.60, the dimes are worth \$2.70, and the quarters are worth \$14.00, making a total of \$17.30.

• PROBLEM 12-71

> Max has \$1.45 in coins. He has fourteen coins in nickels, dimes, and quarters. There are two more nickels than dimes and quarters combined. How many of each kind of coin does he have?

Solution: Let the number of nickels be n, dimes d, and quarters q. The total value of the coins yields one equation.

$$0.05n + 0.10d + 0.25q = 1.45 \qquad (1)$$

The total number of coins yields another.

$$n + d + q = 14 \qquad (2)$$

The number of nickels yields a third.

$$n = d + q + 2 \qquad (3)$$

Substituting the value of n from (3) in (2),

$$d + q + 2 + d + q = 14$$
$$2d + 2q + 2 = 14$$
$$d = 6 - q$$

Substituting this value of d in (2),

$$n + 6 - q + q = 14$$
$$n = 8$$

Substituting $n = 8$ and $d = 6 - q$ in (1),

$$0.05(8) + 0.10(6 - q) + 0.25q = 1.45$$
$$0.40 + 0.60 - 0.10q + 0.25q = 1.45$$
$$0.15q = 0.45$$
$$q = 3$$

Since $d = 6 - q$, $d = 3$. Therefore, Max has $q = 3$ quarters, $d = 3$ dimes, and $n = 8$ nickels.

• PROBLEM 12-72

> The three angles of a triangle are together equal to 180°. The smallest angle is half as large as the largest one, and the sum of the largest and smallest angles is twice the third angle. Find the three angles.

Solution: Let x = the smallest angle,
y = the largest angle,
z = the third angle.

From the given information we formulate the following equations:

$$x + y + z = 180 \qquad (1)$$
$$x = \frac{y}{2} \qquad (2)$$
$$x + y = 2z \qquad (3)$$

We wish to solve for x, y and z. Multiply both sides of equation (2) by 2. Thus, $2x = 2\left(\frac{y}{2}\right)$

$$2x = y$$

Now, subtract y from both sides. Thus,

$$2x - y = y - y,$$

or

$$2x - y = 0 \,. \qquad (4)$$

Subtract $2z$ from both sides of equation (3):

$$x + y - 2z = 2z - 2z \,,$$

or

$$x + y - 2z = 0 \,. \qquad (5)$$

Equation (1) multiplied by 2 gives

$$2x + 2y + 2z = 360 \,. \qquad (6)$$

Equation (5) + (6) gives

$$\begin{array}{r} x + y - 2z = 0 \\ \underline{+\ (2x + 2y + 2z = 360)} \\ 3x + 3y = 360 \end{array} \qquad (7)$$

Equation (4) multiplied by 3 gives

$$6x - 3y = 0 \,. \qquad (8)$$

Equation (7) + (8) gives

$$\begin{array}{r} 3x + 3y = 360 \\ \underline{+\ (6x - 3y = 0)} \\ 9x = 360 \end{array} \;;$$

therefore, $x = 40^\circ$

Substituting 40° for x in (2) gives $40 = \frac{y}{2}$. Multiply both sides by 2: $80^\circ = y$. Substituting 40° for x and 80° for y in (1) gives

$$40^\circ + 80^\circ + z = 180^\circ \,.$$

Subtract 120 from both sides: $z = 60^\circ$.

Thus, the angles are 40°, 80°, and 60°.

Check: Substituting 40° for x, 80° for y, 60° for z gives:

$$40^\circ + 80^\circ + 60^\circ = 180^\circ$$
$$40^\circ = \frac{80^\circ}{2}$$
$$40^\circ + 80^\circ = 2(60^\circ) \,.$$

• PROBLEM 12-73

> Show that the sum of any positive number and its reciprocal cannot be less than 2.

Solution: Express the given example as a mathematical statement, recalling the reciprocal of $x = 1/x$. Write "the sum of any positive number and its reciprocal cannot be less than 2" as

$$a + \frac{1}{a} \not< 2,$$

where a represents the positive number. Thus, the sum of any positive number and its reciprocal can be greater than or equal to 2. We are to show that

$$a + \frac{1}{a} \geq 2, \quad a > 0.$$

If this relation is true, the following inequalities will also hold:

$a^2 + 1 \geqq 2a$, Multiply both members of the inequality by a,

$a^2 - 2a + 1 \geqq 0$, Transpose $2a$

$(a - 1)^2 \geqq 0$, Factor

Now this simple relation is easily shown to be true, for, whether

$a - 1$ is positive, negative, or zero, its square must be non-negative. This is therefore a suitable starting point, and our synthesis, constituting the actual proof, is as follows. Since

$$(a - 1)^2 \geqq 0,$$

for the reason just stated, expansion of the left member gives us the equivalent relations

$a^2 - 2a + 1 \geqq 0$, Expanding

$a^2 + 1 \geqq 2a$, Transposing $2a$

$a + \frac{1}{a} \geqq 2$, Divide both sides of the inequality by a, with $a > 0$.

We see, incidentally, that the equality holds only if $a = 1$; if $0 < a < 1$, or if $a > 1$, the inequality holds.

• PROBLEM 12-74

The sum of three consecutive odd numbers is 45. Find the numbers.

Solution: If we look at a series of odd numbers 1,3,5,7,9,11... we note that each consecutive odd number is 2 more than the one before it, i.e.,

$$3 = 1 + 2$$
$$5 = 3 + 2$$
$$7 = 5 + 2 \quad \text{etc.}$$

Therefore, if x is the first odd number,

$x + 2$ is the second odd number,

and $(x + 2) + 2 = x + (2 + 2) = x + 4$

is the third odd number. Therefore

Let x = the first odd number,

$x + 2$ = the second odd number,

$x + 4$ = the third odd number.

Then, since the sum of these numbers is 45, we have:

$$x + (x + 2) + (x + 4) = 45.$$

Using the Associative and Commutative Laws of Addition:

$$3x + 6 = 45.$$

Subtract 6 from both sides:

$$3x = 45 - 6.$$

Collect terms, $3x = 39.$

Divide by 3, $x = 13$, the first odd number.

Replace x by 13, $x + 2 = 15$, the second odd number.

Replace x by 13, $x + 4 = 17$, the third odd number.

Check: $13 + 15 + 17 = 45,$

$45 = 45.$

• PROBLEM 12-75

The sum of four consecutive even numbers is 140. What are the numbers?

Solution: An even number can be represented by $2n$, where n is an integer. Consecutive even integers (or odd integers) differ by 2. Therefore, four consecutive even integers can be represented by the following:

the first even integer is $2n$,

the next even integer is $2n + 2$

the next even integer is $(2n + 2) + 2 = 2n + 4$, and

the next or fourth even integer is $(2n + 4) + 2 = 2n + 6$.

Hence, the four consecutive even integers are $2n$, $2n + 2$, $2n + 4$, and $2n + 6$. Then

$$2n + (2n + 2) + (2n + 4) + (2n + 6) = 140$$
$$8n + 12 = 140$$
$$8n = 128$$
$$n = 16$$

Therefore:

the first even integer $= 2n = 2(16) = 32$
the next even integer $= 2n + 2 = 2(16) + 2 = 32 + 2 = 34$
the next even integer $= 2n + 4 = 2(16) + 4 = 32 + 4 = 36$
and the next or fourth even integer
$= 2n + 6 = 2(16) + 6 = 32 + 6 = 38$

Note: The sum of these four consecutive even integers =

$$32 + 34 + 36 + 38 = 66 + 36 + 38 = 102 + 38 = 140.$$

• PROBLEM 12-76

The product of two consecutive odd integers is 35. Find the integers.

Solution: Let x represent the first odd integer. Then $x + 2$ represents the next odd integer. The product $x(x + 2)$ equals 35. Thus, we solve the following equation.

$x(x + 2) = 35.$

$x^2 + 2x = 35$ Distributing

$x^2 + 2x - 35 = 0$ Subtracting 35 from both members

$(x + 7)(x - 5) = 0$ Factoring

In order for this product to $= 0$, either $(x + 7) = 0$ or $(x - 5) = 0$. Therefore,

$$x = -7 \quad \text{or} \quad x = 5$$

Since x represents the first odd integer, we see that there are two solutions.

When $x = -7$, $x + 2 = -5$, and $(-7)\cdot(-5) = 35$.
When $x = 5$, $x + 2 = 7$, and $5\cdot 7 = 35$.

The two integers are 5 and 7 or -7 and -5.

• PROBLEM 12-77

> Find 3 consecutive positive integers such that when 5 times the largest be subtracted from the square of the middle one the result exceeds three times the smallest by 7.

Solution: Let x = The smallest number

Then $x + 1$ = The next larger number

And $x + 2$ = The largest number

$$(x + 1)^2 - 5(x + 2) = 3x + 7$$
$$x^2 + 2x + 1 - 5x - 10 = 3x + 7$$
$$x^2 - 6x - 16 = 0$$
$$(x - 8)(x + 2) = 0$$
$$x = -2 \text{ reject}$$

$x = 8$
$x + 1 = 9$
$x + 2 = 10$

Check: $(9)^2 - 5(10) = 3(8) + 7$
$81 - 50 = 24 + 7,\ 31 = 31$

• PROBLEM 12-78

> If the largest of four consecutive odd integers is represented by n, what is the smallest of these integers represented by?

Solution: Smaller consecutive odd integers are obtained by subtracting 2 from each successive odd integer. For example, the sequence 17, 15, 13 may be extended to include 11 and 9.

Hence, if n is the largest of four consecutive odd integers, then the three smaller ones found by subtracting 2 are $(n - 2)$, $(n - 4)$, and $(n - 6)$. The smallest of these is $(n - 6)$.

• PROBLEM 12-79

> Can the sum of three consecutive odd integers be (a) 25? (b) 45?

Solution: Notice that all consecutive odd integers differ by 2:

$$1,\ 1 + 2 = 3,\ 3 + 2 = 5,\ 5 + 2 = 7,\ 7 + 2 = 9,\ \ldots$$

Thus, if we let x = the first consecutive odd integer
$x + 2$ = the 2nd consecutive odd integer
and
$(x+2) + 2 = x + 4$ = the 3rd consecutive odd integer,

(a) We take the sum of these three numbers and determine if it can be 25:

$$x + (x+2) + (x+4) = 25$$
$$3x + 6 = 25$$
$$3x = 19$$
$$x = \frac{19}{3}.$$

Since 19/3 is not an integer, there are no such odd integers.

(b) If 25 is replaced by 45, the equation takes the form

$$x + (x+2) + (x+4) = 45$$
$$3x + 6 = 45$$
$$3x = 39$$
$$x = \frac{39}{3} = 13$$
$$x + 2 = 13 + 2 = 15$$
$$x + 4 = 13 + 4 = 17$$

Thus, the three consecutive odd integers are 13, 15, 17.

In general, if the sum of three consecutive odd integers is to be the number N, then N must be an integral multiple of 3.

AGE AND WORK

• PROBLEM 12-80

> John is 4 times as old as Harry. In six years John will be twice as old as Harry. What are their ages now?

Solution: Let x represent Harry's age now. John's age now is then represented by $4x$. In six years their respective ages will be $(x + 6)$ and $(4x + 6)$. In tabular form our data is:

	Now	In Six Years
John's Age	$4x$	$4x + 6$
Harry's Age	x	$x + 6$

From the statement six years from now John will be twice as old as Harry, we can write the equation necessary to solve the problem. To form an equation from this information we must multiply Harry's age in six years by 2, to

account for John's age being 2 times Harry's. Thus, we have:

$$4x + 6 = 2(x + 6), \text{ or}$$
$$4x + 6 = 2x + 12.$$

We now want to solve for x to obtain Harry's age. To do this we proceed as follows: Subtract 2x from both sides of the equation, $4x + 6 = 2x + 12$. We obtain:

$$4x - 2x + 6 = 2x - 2x + 12$$
$$2x + 6 = 12. \qquad (1)$$

Now, subtracting 6 from both sides of Equation (1), we have:

$$2x + 6 - 6 = 12 - 6$$
$$2x = 6. \qquad (2)$$

Finally, dividing both sides of Equation (2) by 2, we obtain:

$$\frac{2x}{2} = \frac{6}{2} \text{ or, } x = 3.$$

Therefore, Harry's age now is 3 years, and John's is $4(3) = 12$ years.

We can verify these values by observing that in six years Harry will be $(3 + 6)$ or 9 years old and John will be $[4(3) + 6]$ or 18 years old. Therefore, John will then be twice as old as Harry, which was given above.

• PROBLEM 12-81

A mother is now 24 years older than her daughter. In 4 years, the mother will be 3 times as old as the daughter. What is the present age of each?

Solution: Let x = the age of the daughter
Then $x + 24$ = the age of the mother
$x + 4$ = the age of the daughter in 4 years
$x + 24 + 4 = x + 29$ = the age of the mother in 4 years

The relationship used in setting up the equation is:
In 4 years, the mother will be 3 times as old as the daughter.

$$x + 28 = 3(x + 4); \quad x = 8$$

Check: The daughter is now 8 years old and the mother is 32 years old. In 4 years, the daughter will be 12 years old and the mother will be 36 years old. At this time, the mother will be 3 times as old as the daughter.

• PROBLEM 12-82

A man is now 6 times as old as his son. In two years, the man will be 5 times as old as his son. Find the present ages of the man and his son.

Solution: Let x = the present age of the son.
Then let $6x$ = the present age of the father.
$x + 2$ = the son's age in 2 years.
$6x + 2$ = the father's age in 2 years.

The relationship used in setting up the equation is:
In two years the father will be five times as old as the son.

$$6x + 2 = 5(x + 2)$$
$$6x + 2 = 5x + 10$$
$$x = 8$$

Check: The son is now 8 and the father is now 48.
In two years, the son will be 10 and the father will be 50.
At this time, the father will be 5 times as old as the son.

• PROBLEM 12-83

John is now 18 years old and his brother, Charles, is 14 years old. How many years ago was John twice as old as Charles?

Solution: Let x = the number of years ago John was twice as old as Charles.
Then $18 - x$ = John's age x years ago.
And $14 - x$ = Charles' age x years ago.

The relationship used in setting up the equation is:

x years ago, John was twice as old as Charles

$$18 - x = 2(14 - x); \quad x = 10$$

Check: 10 years ago, John was 8 and Charles was 4. At this time, John was twice as old as Charles.

• PROBLEM 12-84

How long will it take Jones and Smith working together to plow a field which Jones can plow alone in 5 hours and Smith alone in 8 hours?

Solution: Here we have

(1) How long will it take Jones and Smith working together to plow a field?
(2) Jones and Smith working together plow a field.
(3) Jones can plow it alone in 5 hours.
(4) Smith can plow it alone in 8 hours.

These sentences become, in turn:

(1') Let x = number of hours that it takes Jones and Smith to plow the field.
(2') (Jones's fractional part of the work) + (Smith's fractional part of the work) = 1, since they do 1 job together
(3') Since Jones does the job alone in 5 hours, we can write 1 job = 5 hours work. Divide both sides of this equation

by 5:

$$\frac{1}{5}\text{ job} = \frac{5 \text{ hours work}}{5}$$

$$\frac{1}{5}\text{ job} = 1 \text{ hour work.} \qquad (1)$$

Hence, Jones does 1/5 of the job in 1 hour. Note that the numerator of the fraction on the left side of equation (1) is equal to the number of hours; i.e., 1 = 1. Therefore, in x hours, Jones does x/5 of the job.

(4') Similarly, in one hour Smith does 1/8 of the job and, in x hours, x/8 of the job. Hence, we have

$$\frac{x}{5} + \frac{x}{8} = 1 \qquad (2)$$

Obtaining a common denominator of 40 for the two fractions on the right side of equation (2):

$$\frac{8(x)}{8(5)} + \frac{5(x)}{5(8)} = 1$$

$$\frac{8x}{40} + \frac{5x}{40} = 1$$

$$\frac{8x + 5x}{40} = 1$$

$$\frac{13x}{40} = 1 \qquad (3)$$

Multiply both sides of equation (3) by 40/13,

$$\frac{\cancel{40}}{\cancel{13}}\left(\frac{\cancel{13}x}{\cancel{40}}\right) = \frac{40}{13} \qquad (1)$$

$x = \frac{40}{13}$ hours $= 3\frac{1}{13}$ hours $\approx$ 3 hours 5 minutes. Therefore, it takes Jones & Smith 3 hours 5 minutesto plow the field.

• PROBLEM 12-85

If A can do a job in 8 days and B can do the same job in 12 days, how long would it take the two men working together?

Solution: Let x = the number of days it would take the two men working together.

Then $\frac{x}{8}$ = the part of the job done by A

and $\frac{x}{12}$ = the part of the job done by B

The relationship used in setting up the equation is:

Part of job done by A + Part of job done by B = 1 job

$$\frac{x}{8} + \frac{x}{12} = 1$$

$$3x + 2x = 24$$

$$x = 4\frac{4}{5} \text{ days}$$

Check:

$$\frac{4\frac{4}{5}}{8} + \frac{4\frac{4}{5}}{12} = 1$$

$$\frac{\frac{24}{5}}{8} + \frac{\frac{24}{5}}{12} = 1$$

$$\frac{3}{5} + \frac{2}{5} = 1$$

• PROBLEM 12-86

A mechanic and his helper can repair a car in 8 hours. The mechanic works 3 times as fast as his helper. How long would it take the helper to make the repair, working alone?

Solution: Let x = Number of hours it would take the mechanic working alone.

Then 3x = Number of hours it would take the helper working alone.

The relationship used in setting up the equation is:

Part of job done by mechanic + Part of job done by helper = 1 job.

$$\frac{8}{x} + \frac{8}{3x} = 1$$

$$x = 10\frac{2}{3} \text{ hours by mechanic}$$

$$3x = 32 \text{ hours by helper .}$$

Check:

$$\frac{8}{10\frac{2}{3}} + \frac{8}{32} = 1$$

$$\frac{24}{32} + \frac{1}{4} = 1$$

$$\frac{3}{4} + \frac{1}{4} = 1$$

• PROBLEM 12-87

A man can do a job in 9 days and his son can do the same job in 16 days. They start working together. After 4 days the son leaves and the father finishes the job alone. How many days did the man take to finish the job alone?

Solution: Let x = the number of days it takes the man to finish the job.

Note that the man actually works (x + 4) days, and the son actually works 4 days.

The relationship used to set up the equation is:

Part of job done by man + Part of job done by boy = 1 job

$$\frac{x+4}{9} + \frac{4}{16} = 1$$

$$16(x+4) + 4(9) = 144$$

$$x = 2\frac{3}{4} \text{ days}$$

Check: $$\frac{2\frac{3}{4}+4}{9} + \frac{4}{16} = 1$$

$$\frac{3}{4} + \frac{1}{4} = 1$$

• PROBLEM 12-88

> A tank can be filled in 9 hours by one pipe, in 12 hours by a second pipe, and can be drained when full, by a third pipe, in 15 hours. How long would it take to fill the tank if it is empty, and if all pipes are in operation?

Solution: Let x = the number of hours the pipes are in operation

Then $\frac{x}{9}$ = part of tank filled by first pipe

and $\frac{x}{12}$ = part of tank filled by second pipe

and $\frac{x}{15}$ = part of tank emptied by third pipe

The relationship used in setting up the equation is:

Part of tank filled by first pipe + Part of tank filled by second pipe - Part of tank emptied by third pipe = 1 Full tank.

$$\frac{x}{9} + \frac{x}{12} - \frac{x}{15} = 1$$

$$20x + 15x - 12x = 180$$

$$x = 7\frac{19}{23} \text{ hours}$$

Check: $$\frac{7\frac{19}{23}}{9} + \frac{7\frac{19}{23}}{12} - \frac{7\frac{19}{23}}{15} = 1$$

$$\left(\frac{180}{23}\cdot\frac{1}{9}\right) + \left(\frac{180}{23}\cdot\frac{1}{12}\right) - \left(\frac{180}{23}\cdot\frac{1}{15}\right) = 1$$

$$\frac{20}{23} + \frac{15}{23} - \frac{12}{23} = 1$$

• PROBLEM 12-89

> Solve the proportion $\frac{x+1}{4} = \frac{15}{12}$.

Solution: Cross multiply to determine x; that is, multiply the numerator of the first fraction by the denominator of the second, and equate this to the product of the numerator of the second and the denominator of the first.

$$(x+1)12 = 4 \cdot 15$$

$$12x + 12 = 60$$

$$x = 4.$$

• PROBLEM 12-90

> If $a/b = c/d$, $a + b = 60$, $c = 3$, and $d = 2$, find b.

Solution: We are given $\frac{a}{b} = \frac{c}{d}$. Cross multiplying we obtain $ad = bc$. Adding bd to both sides, we have $ad + bd = bc + bd$, which is equivalent to $d(a + b) = b(c + d)$ or

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

Replacing $(a + b)$ by 60, c by 3 and d by 2 we obtain

$$\frac{60}{b} = \frac{3+2}{2}$$

$$\frac{60}{b} = \frac{5}{2}.$$

Cross multiplying, $5b = 120$

$$b = 24.$$

• PROBLEM 12-91

> On a map, $\frac{3}{16}$ inch represents 10 miles. What would be the length of a line on the map which represents 96 miles?

Solution: The lengths of line segments on the map are pro-

portional to the actual distances on the earth. If L represents the length of the line segment on the map corresponding to a distance of 96 miles, then

$$\frac{\frac{3}{16}\text{ inches}}{L\text{ inches}} = \frac{10\text{ miles}}{96\text{ miles}}$$

$$\frac{3}{16}(96) = 10L \qquad \text{by cross multiplying}$$

$$L = \frac{(3)(96)}{(16)(10)} = \frac{3(6)}{10}$$

$$L = \frac{18}{10}$$

$$L = 1\frac{4}{5}\text{ inches.}$$

• PROBLEM 12-92

> Find the height of a tree which casts a shadow 20 feet long at the same time a vertical yard stick casts a shadow 30 inches long.

Solution: Set up this proportion by realizing that each height is proportional to the length of its shadow. Since this is the case, the ratio of the height of the tree and its shadow can be equated with the ratio of the height of the yardstick and its shadow.

Let: H represent the height of the tree
h represent the height of the yardstick = 36 in
S represent the length of the shadow of the tree = 20 ft
s represent the length of the shadow of the yardstick = 30 in.

We now have: $\frac{H}{S} = \frac{h}{s}$

This equation is solved by multiplying both sides of the Least Common Denominator, LCD, which is found by multiplying the two denominators together.

$$LCD = Ss$$

Our equation becomes

$$Ss\left(\frac{H}{S}\right) = Ss\left(\frac{h}{s}\right), \text{ or } Hs = hS.$$

Dividing by s: $H = \frac{hS}{s}$

$$H = \frac{36\text{ in X } 20\text{ ft}}{30\text{ in}}\text{ ; inches cancel; and}$$

$$H = \frac{6}{5}\text{ X } 20\text{ ft} = 24\text{ ft.}$$

Hence, the height of the tree is 24 feet.

• PROBLEM 12-93

> Find the ratios of x : y : z from the equations
>
> $7x = 4y + 8z, \quad 3z = 12x + 11y.$

Solution: By transposition we have

$$7x - 4y - 8z = 0$$
$$12x + 11y - 3z = 0.$$

To obtain the ratio of x : y we convert the given system into an equation in terms of just x and y. z may be eliminated as follows: Multiply each term of the first equation by 3, and each term of the second equation by 8, and then subtract the second equation from the first. We thus obtain:

$$\begin{array}{r} 21x - 12y - 24z = 0 \\ \underline{-\,(96x + 88y - 24z = 0)} \\ -75x + 100y = 0 \end{array}$$

Dividing each term of the last equation by 25 we obtain,

$$-3x - 4y = 0 \quad \text{or,}$$
$$-3x = 4y.$$

Dividing both sides of this equation by 4, and by − 3, we have the proportion:

$$\frac{x}{4} = \frac{y}{-3}.$$

We are now interested in obtaining the ratio of y : z. To do this we convert the given system of equations into an equation in terms of just y and z, by eliminating x as follows: Multiply each term of the first equation by 12, and each term of the second equation by 7, and then subtract the second equation from the first. We thus obtain:

$$84x - 48y - 96z = 0$$

$$\underline{-(84x + 77y - 21z = 0)}$$

$$-125y - 75z = 0.$$

Dividing each term of the last equation by 25 we obtain

$$-5y - 3z = 0, \quad \text{or,}$$

$$-3z = 5y$$

Dividing both sides of this equation by 5, and by -3, we have the proportion:

$$\frac{z}{5} = \frac{y}{-3}.$$

From this result and our previous result we obtain:

$\frac{x}{4} = \frac{y}{-3} = \frac{z}{5}$ as the desired ratios.

● PROBLEM 12-94

If $(2ma + 6mb + 3nc + 9nd)(2ma - 6mb - 3nc + 9nd)$
$= (2ma - 6mb + 3nc - 9nd)(2ma + 6mb - 3nc - 9nd)$,
prove that a, b, c, d are proportionals.

Solution: Dividing both sides of the given equation by $(2ma - 6mb - 3nc + 9nd)$, and then by $(2ma - 6mb + 3nc - 9nd)$ we have

$$\frac{2ma + 6mb + 3nc + 9nd}{2ma - 6mb + 3nc - 9nd} = \frac{2ma + 6mb - 3nc - 9nd}{2ma - 6mb - 3nc + 9nd}.$$

Since the above two ratios are of the form $\frac{a}{b} = \frac{c}{d}$, we can use the Law of Proportions which states that

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Doing this we obtain,

$$\frac{2ma + 6mb + 3nc + 9nd + (2ma - 6mb + 3nc - 9nd)}{2ma + 6mb + 3nc + 9nd - (2ma - 6mb + 3nc - 9nd)} =$$

$$\frac{2ma + 6mb - 3nc - 9nd + (2ma - 6mb - 3nc + 9nd)}{2ma + 6mb - 3nc - 9nd - (2ma - 6mb - 3nc + 9nd)}$$

or, $\frac{4ma + 6nc}{12mb + 18nd} = \frac{4ma - 6nc}{12mb - 18nd}$; and factoring gives us,

$$\frac{2(2ma + 3nc)}{2(6mb + 9nd)} = \frac{2(2ma - 3nc)}{2(6mb - 9nd)} \quad \text{or,}$$

$$\frac{2ma + 3nc}{6mb + 9nd} = \frac{2ma - 3nc}{6mb - 9nd}.$$

Now, since $\frac{a}{b} = \frac{c}{d}$ can be alternately written as $\frac{a}{c} = \frac{b}{d}$, we write:

$$\frac{2ma + 3nc}{2ma - 3nc} = \frac{6mb + 9nd}{6mb - 9nd}.$$

Now, rewriting this last proportion as,

$$\frac{2ma + 3nc + (2ma - 3nc)}{2ma + 3nc - (2ma - 3nc)} = \frac{6mb + 9nd + (6mb - 9nd)}{6mb + 9nd - (6mb - 9nd)}$$

we obtain: $\frac{4ma}{6nc} = \frac{12mb}{18nd}.$

Again, using the fact that $\frac{a}{b} = \frac{c}{d}$ can be rewritten as $\frac{a}{c} = \frac{b}{d}$, we write:

$$\frac{4ma}{12mb} = \frac{6nc}{18nd} \quad \text{or,}$$

$$\frac{a}{3b} = \frac{c}{3d}.$$

Thus, $\frac{a}{b} = \frac{c}{d}$ or, $a : b = c : d$.

● PROBLEM 12-95

If y varies directly with respect to x and $y = 3$ when $x = -2$, find y when $x = 8$.

Solution: If y varies directly as x then y is equal to some constant k times x; that is, $y = kx$ where k is a constant. We can now say $y_1 = kx_1$ and $y_2 = kx_2$ or $\frac{y_1}{x_1} = k$, $\frac{y_2}{x_2} = k$ which implies $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ which is a proportion.

We use the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. Thus $\frac{3}{-2} = \frac{y_2}{8}$. Now solve for y_2:

$$8\left(\frac{3}{-2}\right) = 8\left(\frac{y_2}{8}\right)$$

$$-12 = y_2.$$

When $x = 8$, $y = -12$.

• PROBLEM 12-96

According to Hooke's Law, the length of a spring, S, varies directly as the force, F, applied on the spring. In a spring to which Hooke's Law applies, a force of 18.6 lb stretches the spring by 1.27 in. Find k, the proportionality constant.

Solution: The direct variation of the length of the spring, S, and the force applied on it, F, is expressed symbolically as

$F = kS$, where k is the constant of proportionality.

We are given that $F = 18.6$ lb and $S = 1.27$ in. Thus, it is necessary merely to substitute the given values in the equation $F = kS$, obtaining

$$18.6 = k(1.27),$$

from which $k = 18.6/1.27 = 14.65$ lb/in.

• PROBLEM 12-97

The area of a sphere is directly proportional to the square of the radius. If a sphere of radius 3 inches has an area of 36π square inches, deduce the formula for the area of a sphere.

Solution: If a quantity x is directly proportional to another quantity y, x is the product of a constant k and y; that is, $x = ky$. Let a = the area of a sphere and let r = the radius of the sphere. Since the area a of a sphere is directly proportional to the square of the radius r, the following equation can be written:

$$a = kr^2 \qquad (1)$$

When the area a is 36π square inches, the radius is 3 inches. Then, using equation (1):

$$36\pi = k(3)^2$$

$$36\pi = k(9)$$

$$36\pi = 9k$$

Divide both sides of this equation by 9:

$$\frac{36\pi}{9} = \frac{9k}{9}$$

$$4\pi = k$$

Substituting this value for the constant k in equation (1): $a = (4\pi)r^2$, is the equation which represents the formula for the area of a sphere.

• PROBLEM 12-98

A falling body strikes the ground with a velocity v which varies directly as the square root of the distance s it falls. If a body that falls 100 feet strikes the ground with a velocity of 80 feet per second, with what velocity will a ball dropped from the Washington monument (approximately 550 feet high) strike the ground?

Solution: A variable y is said to vary directly as another variable x if y is equal to some constant k times x, that is, if $y = kx$. Here, the variable v, velocity, varies directly as the square root of the distance s. The general equation for this variation is

$$v = k\sqrt{s} \qquad (1)$$

Since $v = 80$ when $s = 100$, we have

$$80 = k\sqrt{100} \qquad (2)$$

$$80 = k(10)$$

$$k = \frac{80}{10}$$

$$k = 8. \qquad (3)$$

Replacing $k = 8$ in Equation 1

$$v = 8\sqrt{s}. \qquad (4)$$

When $s = 550$

$$v = 8\sqrt{550} \qquad (5)$$

$$\cong 8(23.45) \cong 188.$$

Therefore the velocity is approximately 188 feet per second.

• **PROBLEM 12-99**

The resistance R of a given size of wire at constant temperature varies directly as the length ℓ. It is found that the resistance of 100 feet of number 14 copper is 0.253 ohm. Construct a table of values for the given lengths of number 14 copper wire assuming the temperature is constant.

ℓ	25	75	125	175	225
R					

Solution: Direct variation implies a variable y is equal to a constant c times x; that is, $y = cx$. In this particular example, the resistance R varies directly as the length ℓ. Since several values of R are to be found, we use the general equation for this variation. Thus,

$$R = k\ell \tag{1}$$

Since $R = 0.253$ when $\ell = 100$

$$0.253 = 100k \tag{2}$$

Solving for k

$$k = 0.00253 \tag{3}$$

The specific equation is therefore

$$R = 0.00253\ell \tag{4}$$

The values of R corresponding to the given values of ℓ may then be found directly from Equation 4, as follows.

ℓ	25	75	125	175	225
R	0.063	0.190	0.316	0.443	0.569

Notice that if values of ℓ were to be determined from given values R, it would be convenient to solve Equation 4 for ℓ in terms of R. Thus

$$\ell = \frac{R}{0.00253} \tag{5}$$

or

$$\ell = 395.26R$$

• **PROBLEM 12-100**

The square of the time of a planet's revolution varies as the cube of its distance from the Sun; find the time of Venus' revolution, assuming the distances of the Earth and Venus from the Sun to be 91 ¼ and 60 millions of miles, respectively.

Solution: Since x varies directly as y, $x = ky$, where k is a constant, called the constant of proportionality, or the constant of variation. We are given that

$$(\text{time of revolution})^2 = k\ (\text{distance from Sun})^3.$$

Let P be the periodic time measured in days, D the distance in millions of miles; we have

$$P^2 = kD^3,$$

where k is some constant.

We are given D for the Earth, and since we know that P for Earth is 365 days, we can use this information to find k. We have:

$(365)^2 = k(91\tfrac{1}{4})^3$. This can be rewritten as:

$$365 \times 365 = k \times 91\tfrac{1}{4} \times 91\tfrac{1}{4} \times 91\tfrac{1}{4}$$

Now, since $91\tfrac{1}{4} = \frac{365}{4}$, we write:

$$365 \times 365 = k\left(\frac{365}{4} \times \frac{365}{4} \times \frac{365}{4}\right) \quad \text{or,}$$

$$365 \times 365 = k\left(\frac{365 \times 365 \times 365}{4 \times 4 \times 4}\right).$$ Solving for k we have:

$$\left(\frac{4 \times 4 \times 4}{365 \times 365 \times 365}\right) 365 \times 365 = k. \text{ Thus,}$$

$$k = \frac{4 \times 4 \times 4}{365}.$$

We can now use this value of k to solve for the time of Venus' revolution. We are given that $D = 66$. Thus, we have:

$P^2 = kD^3$ and, substituting we obtain:

$$P^2 = \left(\frac{4 \times 4 \times 4}{365}\right)(66)^3; \text{ therefore,}$$

P = the time of Venus' revolution =

$$\sqrt{\frac{4 \times 4 \times 4}{365} \cdot (66)^3}$$

• **PROBLEM 12-101**

Express y in terms of x if y is inversely proportional to x and $y = 2$ when $x = 3$.

Solution: y is inversely proportional to x means there exists a number k such that $y = k/x$. Our problem will be solved when we determine the number k. If we substitute 3 for x and 2 for y in the equation $y = k/x$, we find $2 = k/3$. Multiplying both sides by 3 we obtain $k = 6$. Thus, the equation relating x and y is $y = 6/x$.

• **PROBLEM 12-102**

If y varies inversely as the cube of x, and y = 7 when x = 2, express y as a function of x.

Solution: The relationship "y varies inversely with respect to x" is expressed as,

$$y = \frac{k}{x}.$$

The inverse variation is now with respect to the cube of x, x^3 and we have,

$$y = \frac{k}{x^3}$$

Since y = 7 and x = 2 must satisfy this relation, we replace x and y by these values,

$$7 = \frac{k}{2^3} = \frac{k}{8},$$

and we find $k = 7 \cdot 8 = 56$. Substitution of this value of k in the general relation gives,

$$y = \frac{56}{x^3},$$

which expresses y as a function of x. We may now, in addition find the value of y corresponding to any value of x. If we had the added requirement to find the value of y when x = 1.2, x = 1.2 would be substituted in the function so that for x = 1.2, we have,

$$y = \frac{56}{(1.2)^3} = \frac{56}{1.728} = 32.41$$

Other expressions in use are "is proportional to" for "varies directly," and "is inversely proportional to" for "varies inversely."

• **PROBLEM 12-103**

The cube root of x varies inversely as the square of y; if x = 8 when y = 3, find x when y = 1½.

Solution: If x varies inversely as y, then $x = \frac{m}{y}$, and in this case $\sqrt[3]{x}$ varies inversely as y^2, thus,

$$\sqrt[3]{x} = \frac{m}{y^2}, \text{ where m is constant.}$$

Putting x = 8, y = 3, we have $\sqrt[3]{8} = \frac{m}{3^2}$ or, $2 = \frac{m}{9}$; therefore, m = 18.

Now, we want to find x when $y = 1\frac{1}{2} = \frac{3}{2}$. We know from above that m = 18, and since $\sqrt[3]{x} = \frac{m}{y^2}$, by substitution we obtain:

$$\sqrt[3]{x} = \frac{18}{\left(\frac{3}{2}\right)^2}$$

$$\sqrt[3]{x} = \frac{18}{\frac{9}{4}}$$

$$\sqrt[3]{x} = 18\left(\frac{4}{9}\right) = 8. \text{ Now cubing both sides we obtain:}$$

$$x = 8^3 \text{ or } x = 512.$$

• **PROBLEM 12-104**

The weight W of an object above the earth varies inversely as the square of the distance d from the center of the earth. If a man weighs 180 pounds on the surface of the earth, what would his weight be at an altitude of 1000 miles? Assume the radius of the earth to be 4000 miles.

Solution: W varies inversely with d^2; therefore $W = \frac{k}{d^2}$ where k is the proportionality constant. Similarly, $W_1 = \frac{k}{d_1^2}$, $W_2 = \frac{k}{d_2^2}$ and, solving these two equations for k,

$W_1 d_1^2 = k$ and $W_2 d_2^2 = k$. Hence,

$$k = W_1 d_1^2 = W_2 d_2^2$$

$$\text{or } \frac{W_1 d_1^2}{W_2} = \frac{\cancel{W_2} d_2^2}{\cancel{W_2}}$$

$$\frac{W_1 d_1^2}{W_2} = d_2^2$$

$$\frac{W_1 \cancel{d_1^2}}{W_2 \cancel{d_1^2}} = \frac{d_2^2}{d_1^2}$$

$$\frac{W_1}{W_2} = \frac{d_2^2}{d_1^2} \qquad (1)$$

Letting d_1 = radius of the earth, 4000, then d_2 = 4000 + 1000 = 5000.

Substituting the given values in Equation (1):

$$\frac{180}{W_2} = \frac{5000^2}{4000^2} = \frac{(5 \times 1000)^2}{(4 \times 1000)^2} = \frac{5^2 \times \cancel{1000^2}}{4^2 \times \cancel{1000^2}}$$

$$= \frac{5^2}{4^2}$$

$$= \frac{25}{16}$$

$$\frac{180}{W_2} = \frac{25}{16}$$

$$\cancel{W_2}\left(\frac{180}{\cancel{W_2}}\right) = W_2\left(\frac{25}{16}\right)$$

$$180 = \frac{25}{16}W_2$$

$$\frac{16}{\cancel{25}_{5}}\left(\overset{36}{\cancel{180}}\right) = \frac{\cancel{16}}{\cancel{25}}\left(\frac{\cancel{25}}{\cancel{16}}W_2\right)$$

$$\frac{576}{5} = W_2$$

$$115\tfrac{1}{5}\text{ pounds} = W_2$$

$$\text{or } 115.2\text{ pounds} = W_2$$

• PROBLEM 12-105

If y varies jointly as x and z, and 3x:1 = y:z, find the constant of variation.

Solution: A variable s is said to vary jointly as t and v if s varies directly as the product tv; that is, if s = ctv where c is called the constant of variation.

Here the variable y varies jointly as x and z with k as the constant of variation.

$$y = kxz$$

$$3x:1 = y:z$$

Expressing these ratios as fractions.

$$\frac{3x}{1} = \frac{y}{z}$$

Solving for y by cross-multiplying,

$$y = 3xz$$

Equating both relations for y we have:

$$kxz = 3xz$$

Solving for the constant of variation, k, we divide both sides by xz, $k = 3$

• PROBLEM 12-106

The pressure of wind on a sail varies jointly as the area of the sail and the square of the wind's velocity. When the wind is 15 miles per hour, the pressure on a square foot is one pound. What is the velocity of the wind when the pressure on a square yard is 25 pounds?

Solution: Let p = pressure of the wind, in pounds
v = the velocity of the wind, in miles per hour
a = the area of the sail, in square feet.

Pressure, p, varies jointly as the area of the sail, a, and the square of the wind's velocity, v^2. Therefore p varies directly as the product av^2 times a proportionality constant, k. k must be determined before we can proceed to find v as desired. Use the given information a = 1 and p = 1 when v = 15 to determine the proportionality constant, k.

$$p = kav^2.$$

$$1 = k(1)(225).$$

$$k = \frac{1}{225}, \text{ value of the proportionality constant.}$$

Now we can find v using $k = \frac{1}{225}$ when p = 25 and a = 9 (1 yard = 3 feet, 1 square yard = 9 square feet).

$$p = \frac{1}{225}av^2.$$

$$25 = \frac{1}{225}(9)v^2.$$

$$v^2 = \frac{(25)(225)}{9}$$

$$v = \sqrt{\frac{(25)(225)}{9}} = \frac{(5)(15)}{3}$$

$$v = 25, \text{ number of miles per hour.}$$

• PROBLEM 12-107

The elongation, E, of a steel wire when a mass, m, is hung from its free end varies jointly as m and the length, x, of the wire and inversely as the cross sectional area A, of the wire. Given that E = 0.001 inches when m = 20 pounds, x = 10 inches, and A = 0.01 square inches, find E when m = 40 pounds, x = 15.5 inches, and A = 0.015 square inches.

Solution: If E is directly proportional to m and x and inversely proportional to A, our equation is

$$E = \frac{kmx}{A}$$

where k is a constant which we can determine from the given information.

Since, when m = 20, x = 10, and A = .01, E = .001, we can use this information to solve for k. Substituting we obtain:

$$0.001 = \frac{k(20)(10)}{.01}$$

, and multiplying both sides by $\frac{.01}{(20)(10)}$ we obtain:

$$k = \frac{(.001)(.01)}{200} = \frac{.00001}{200} .$$

Multiplying numerator and denominator by 100,000 (move the decimal 5 places to the right) we obtain:

$$\frac{1}{20,000,000}$$

Hence, $$E = \frac{1}{20,000,000}\frac{mx}{A}$$

and when m = 40, x = 15.5, and A = 0.015, we have:

$$E = \frac{1}{20,000,000}\frac{40(15.5)}{0.015}$$

$$= 0.00207 \text{ inches (approximately).}$$

• PROBLEM 12-108

The current, i, in amperes in an electric circuit varies directly as the electromotive force, E, in volts and inversely as the resistance, R, in ohms. If, in a certain circuit, i = 30 amperes when R = 15 ohms and E = 450 volts, find i when R = 20 ohms and E = 200 volts.

Solution: The relation "i varies directly as E and inversely as R" may be expressed as $i = \frac{kE}{R}$ where k is the proportionality constant. k must be determined before i can be found as desired.

i = kE/R and so

$$30 = \frac{k \cdot 450}{15} \quad \text{or, } k = 30\left(\frac{15}{450}\right)$$
$$= \frac{15}{15}$$
$$= 1.$$

Hence i = E/R and when R = 20 and E = 200, we have i = = 200/20 = 10 amperes.

• PROBLEM 12-109

A certain beam L ft. long has a rectangular cross section b in. in horizontal width and d in. in vertical depth. It is found that, when the beam is supported at the ends, the deflection D at the center varies directly as the fourth power of L, inversely as b, and inversely as the cube of d. If the length is decreased by 10 per cent but the width kept the same, by how much should the depth be changed in order that the same deflection D be obtained?

Solution: A quantity m varies directly as another q quantity n if m equals the product of a constant and n; that is, m = cn where c is the constant. Also, a quantity p varies inversely as another quantity q if p equals the product of a constant and the reciprocal of q; that is, p = c(1/q) = c/q where c is the constant. From the statement of the problem, we see that the combined variation is given by

$$D = \frac{kL^4}{bd^3} \text{, where } k \text{ is a constant.}$$

Since corresponding values of the four variables are not known, we cannot determine the value of the constant of proportionality k. But if one set of variables is designated with the subscript 1, and the new set with the subscript 2, we have

$$D_1 = \frac{kL_1^4}{b_1d_1^3} , \quad D_2 = \frac{kL_2^4}{b_2d_2^3} .$$

Since it is desired to obtain a relationship between d_1 and d_2, carry out the following procedure in order to isolate d_1 and d_2 on different sides of the equation.

Divide D_2 by D_1:

$$\frac{D_2}{D_1} = \frac{kL_2^4/b_2d_2^3}{kL_1^4/b_1d_1^3} \tag{1}$$

Since division by a fraction is equivalent to multiplication by that fraction's reciprocal, equation (1) becomes:

$$\frac{D_2}{D_1} = \frac{kL_2^4}{b_2d_2^3} \cdot \frac{b_1d_1^3}{kL_1^4} = \frac{L_2^4b_1d_1^3}{b_2d_2^3L_1^4} = \frac{L_2^4b_1d_1^3}{L_1^4b_2d_2^3}$$

or

$$\frac{D_2}{D_1} = \frac{L_2^4b_1d_1^3}{L_1^4b_2d_2^3}$$

In this problem we have $D_2 = D_1$, $L_2 = 0.9L_1$, and $b_2 = b_1$. Therefore we get

$$\frac{D_1}{D_1} = \frac{(0.9L_1)^4b_1d_1^3}{L_1^4b_1d_2^3}$$

$$1 = \frac{(0.9)^4\cancel{L_1^4}\cancel{b_1}d_1^3}{\cancel{L_1^4}\cancel{b_1}d_2^3}$$

$$1 = (0.9)^4\frac{d_1^3}{d_2^3}$$

Multiplying both sides by d_2^3:

$$d_2^3(1) = \cancel{d_2^3}(0.9)^4\frac{d_1^3}{\cancel{d_2^3}}$$

$$d_2^3 = (0.9)^4 d_1^3$$

Take the cube root of each side:

$$\sqrt[3]{d_2^3} = \sqrt[3]{(0.9)^4d_1^3}$$

$$d_2 = \sqrt[3]{(0.9)^4}\ \sqrt[3]{d_1^3}$$

$$d_2 = \sqrt[3]{(0.9)^4}\ d_1$$

$$d_2 = (0.9)^{4/3}\ d_1$$

$$d_2 = \left[(0.9)^{1/3}\right]^4 d_1$$

$$d_2 = \left[\sqrt[3]{0.9}\right]^4 d_1$$

$$d_2 \approx (.966)^4\ d_1$$

$$d_2 \approx .871\ d_1$$

Hence, the new depth, d_2 is only $.871\ d_1$. Subtracting $.871\ d_1$ from $1\ d_1$:

$$\begin{array}{r} 1.000\ d_1 \\ -\ 0.871\ d_1 \\ \hline 0.129\ d_1 \end{array}$$

Therefore, the depth has been decreased by an amount of .129 or

$$\frac{129}{1000} = \frac{129}{10} \times \frac{1}{100}$$

$$= \frac{129}{10}\ \%$$ since "hundredths means per cent"

$$= 12.9\%.$$

Hence, the depth has been decreased approximately 13%.

COSTS

• PROBLEM 12-110

Reserved seat tickets to a football game are \$6 more than general admission tickets. Mr. Jones finds that he can buy general admission tickets for his whole family of five for only \$3 more than the cost of reserved seat tickets for himself and Mrs. Jones. How much do the general admission tickets cost?

Solution: Let x = the cost of general admission tickets.

Let $x + 6$ = the cost of reserved seat tickets.

Thus, $5x$ is the cost of five general admission tickets and $2(x+6)$ is the cost of two reserved seat tickets. Since the five tickets cost three dollars more than the two reserved tickets,

$$\begin{aligned} 5x &= 2(x + 6) + 3 \\ 5x &= 2x + 12 + 3 \\ 5x &= 2x + 15 \\ 3x &= 15 \\ x &= 5, \end{aligned}$$

and $$x + 6 = 5 + 6 = 11$$

Thus, general admission tickets are \$5.00, and reserve tickets are \$11.00.

• PROBLEM 12-111

A television set cost a dealer \$102. At what price should he mark the set so that he can give a discount of 15% from the marked price and still make a profit of 20% on the selling price?

Solution: Let x = The marked price

Then $.85x$ = The selling price

The relationship used to set up the equation is:

Selling price = Cost + Profit

$$.85x = 102 + (.20)(.85x)$$
$$.85x = 102 + .17x$$
$$.68x = 102$$
$$x = \$150$$

Check: The marked price = \$150. The selling price is 15% less or \$127.50. Since the cost is \$102, the profit is \$25.50. The profit (\$25.50) is 20% of the selling price (\$127.50).

• PROBLEM 12-112

> At a movie showing there were 356 paid admissions. The total receipts were \$287.40. If orchestra seats sold for \$.90 and balcony seats for \$.65, how many of each kind were sold?

Solution: Let x = The number of \$.90 seats sold

Then $(356 - x)$ = The number of \$.65 seats sold

The relationship used to set up the equation is:

Orchestra receipts + Balcony receipts = Total receipts or \$287.40

$$.90x + .65(356 - x) = 287.40$$
$$90x + 65(356 - x) = 287.40$$
$$90x + 23140 - 65x = 287.40$$
$$25x = 5600$$
$$x = 224 \text{ orchestra seats sold}$$
$$356 - x = 132 \text{ balcony seats sold}$$

Check: 224 at \$.90 = \$201.60
132 at \$.65 = \$85.80
356 tickets for \$287.40

• PROBLEM 12-113

> A merchant paid \$1,800 for a group of men's suits. He sold all but 5 of the suits at \$20 more per suit than he paid, thereby making a profit of \$200 on the transaction. How many suits did the merchant buy?

Solution: Let x = The number of suits the merchant bought

Then $\frac{1,800}{x}$ = The cost of each suit

The relationship used to set up the equation is:

The number of suits sold X The selling price of each suit = \$1,800 + \$200

$$(x - 5)\left(\frac{1,800}{x} + 20\right) = 2,000$$
$$1,800 + 20x - \frac{9,000}{x} - 100 = 2,000$$
$$20x - \frac{9,000}{x} = 300$$
$$20x^2 - 300x - 9,000 = 0$$
$$x^2 - 15x - 450 = 0$$
$$(x - 30)(x + 15) = 0$$
$$x = 30, \quad x = -15 \text{ reject}$$

The merchant bought 30 suits.

Check: The merchant bought 30 suits at \$60 each. He sold 25 suits at \$80 each, thus taking in \$2,000.

• PROBLEM 12-114

> A dealer can buy a certain number of ties for \$30. If 5 more could be bought for the same money, the price would be \$3.60 less per dozen. What is the price per dozen?

Solution: Let x = The number of ties the dealer can buy for \$30.

Then $\frac{30}{x}$ = The cost per tie

And $\frac{30}{x+5}$ = The supposed cost per tie

The relationship used to set up the equation is:
The supposed cost per dozen = The old cost per dozen minus \$3.60

$$12\left(\frac{30}{x+5}\right) = 12\left(\frac{30}{x}\right) - \frac{18}{5}$$
$$12(30)(5x) = 12(30)(5)(x + 5) - 18(x)(x + 5)$$
$$1,800x = 1,800x + 9,000 - 18x^2 - 90x$$
$$18x^2 + 90x - 9,000 = 0$$
$$x^2 + 5x - 500 = 0$$
$$(x + 25)(x - 20) = 0$$
$$x = -25 \text{ reject}, \quad x = 20$$

The man bought 20 ties for \$30, which is at the rate of \$18 per dozen.

Check: The man bought 20 ties for \$30, paying \$1.50 per tie or \$18 per dozen. Had he bought 25 ties for \$30 he would have paid \$1.20 per tie or \$14.40 per dozen. This would have been a saving of \$3.60 per dozen.

• PROBLEM 12-115

A haberdasher sold 3 shirts and 4 ties to one customer for \$18.70. Another customer bought 4 shirts and 7 ties of the same quality for \$27.75. What was the price per shirt and price per tie?

Solution: Let x = Cost of one shirt
And y = Cost of one tie

(1) $3x + 4y = 1870$
(2) $4x + 7y = 2775$

When the equations are solved simultaneously it is found that $x = 398$, i.e., the price of a shirt is \$3.98 and $y = 169$, i.e., the price of a tie is \$1.69.

• PROBLEM 12-116

A real estate dealer received \$1,200 in rents on two dwellings last year, and one of the dwellings brought \$10 per month more than the other. Find the monthly rental on each if the more expensive house was vacant for 2 months.

Solution: On inspecting the problem we see that there are two basic relations involved, the relation between the separate rentals, and the relation between the monthly rentals and the income per year. Since the monthly rentals differ by \$10, we let

x = monthly rental of the more expensive house in dollars
y = monthly rental of the less expensive house in dollars

and we write $\quad x = y + 10$

or $\quad x - y = 10 \qquad (1)$

Furthermore, since the first of the two houses was rented

for 10 months and the other was rented for 12 months, we know that $10x + 12y$ is the total annual income. Hence

$$10x + 12y = 1,200 \qquad (2)$$

We now solve Eq. (1) and (2) simultaneously by eliminating y:

$12x - 12y = 120$	multiplying (1) by 12
$\underline{10x + 12y = 1200}$	(2) recopied
$22x = 1,320$	equating the sums of corresponding members
$x = 60$	solving for x
$60 - y = 10$	replacing x by 60 in (1)
$y = 50$	solving for y

Therefore, the solution set $\{(x, y)\} = \{(60, 50)\}$, and it follows that the monthly rentals are \$60 and \$50, respectively.

We check the obtained values by substituting in equations (1) and (2). Thus

$$x - y = 10 \qquad (1)$$
$$60 - 50 = 10$$
$$10 = 10$$

$$10x + 12y = 1200 \qquad (2)$$
$$10(60) + 12(50) = 1200$$
$$600 + 600 = 1200$$
$$1200 = 1200$$

• PROBLEM 12-117

The income of a business is at the rate of \$1000 a week and the expenses are at the rate of \$900 a week. If the business was worth \$8000 at the start, obtain the function showing the value of the business at the end of t weeks, assuming that the profits are retained in the business.

Solution: Since income exceeds expenses by \$100 a week,

the rate of increase of the function is $m = 100$; the value of the function at $t = 0$ is 8000, that is, $b = f(0) = 8000$, since the original value was 8000. If t represents the number of weeks, the desired function of time will be linear with slope 100 and y-intercept 8000; and $f(t) = 100t + 8000$.

The time at which the business is worth a specified amount is of particular interest. For example, the time at which the business is worth $10,000 is seen to be $t = 20$ weeks, for, with $t = 20$, we have

$$100 \cdot 20 + 8000 = 10{,}000$$

This value of t may also be thought of as the value of t for which the function

$$(100t + 8000) - 10{,}000, \quad \text{or } 100t - 2000$$

vanishes.

More generally, if we set the function $mx + b$ equal to c, we get

$$mx + b = c, \quad \text{or} \quad mx + b - c = 0.$$

The value of x which satisfies the equation in either of the two forms is then $x = (c - b)/m$. From the second form of the equation it is seen that for this value of x the value of the function $mx + b - c$ is zero.

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